

# PARAMETRICES AND EXACT PARALINEARISATION OF SEMI-LINEAR BOUNDARY PROBLEMS

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**ABSTRACT.** The subject is parametrices for semi-linear problems, based on parametrices for linear boundary problems and on non-linearities that decompose into solution-dependent linear operators acting on the solutions. Non-linearities of product type are shown to admit this via exact parilinearisation. The parametrices give regularity properties under weak conditions; improvements in sub-domains result from pseudo-locality of type 1, 1-operators. The framework encompasses a broad class of boundary problems in Hölder and  $L_p$ -Sobolev spaces (and also Besov and Lizorkin–Triebel spaces). The Besov analyses of homogeneous distributions, tensor products and halfspace extensions have been revised. Examples include the von Karman equation.

## 1. INTRODUCTION

This article presents a parametrix construction for semi-linear boundary problems as well as the resulting regularity properties in  $L_p$ -Sobolev spaces. The work is based on investigations of pseudo-differential boundary operators, paramultiplication and function spaces of J.-M. Bony, G. Grubb, V. Rychkov and the author [Bon81, Gru95, Ryc99b, Joh95, Joh96]; it is also inspired by joint work with T. Runst [JR97] on solvability of semi-linear problems.

Assume eg that  $A$  is an elliptic differential operator, that  $\{A, T\}$  is a linear elliptic boundary problem on a domain  $\Omega \subset \mathbb{R}^n$  and that, for a suitable non-linear operator  $Q$ , the function  $u$  is a given solution of the problem

$$Au + Q(u) = f \quad \text{in } \Omega, \quad Tu = \varphi \quad \text{on } \partial\Omega. \quad (1.1)$$

It is then a main point to establish a family of parametrices  $P_u^{(N)}$ ,  $N \in \mathbb{N}$ , that are linear operators yielding the following new formula for  $u$ :

$$u = P_u^{(N)}(Rf + K\varphi + \mathcal{R}u) + (RL_u)^N u. \quad (1.2)$$

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Here  $({}_R K)$  is a left-parametrix of the linear problem, ie  $({}_R K)(\begin{smallmatrix} A \\ T \end{smallmatrix}) = I - \mathcal{R}$  where  $\mathcal{R}$  has range in  $C^\infty(\overline{\Omega})$ , while  $L_u$  is an exact parilinearisation of  $Q(u)$ . In (1.2),  $P_u^{(N)}$  has order zero and can roughly be seen as a modifier of data's contribution to  $u$ , while  $(RL_u)^N$  is an error term analogous to the negligible errors in pseudo-differential calculi; it can have any finite degree of smoothness by choosing  $N$  large enough. Precise assumptions on  $\{A, T\}$  and especially  $Q$  will follow further below.

The motivation was partly to provide an alternative to boot-strap arguments, for in the general  $L_p$ -setting these can require somewhat lengthy descriptions, even though the strategy is clear. It was also hoped to find purely analytical proofs, without iteration, of the regularity properties.

These goals are achieved with the parametrix formula (1.2), for the regularity of  $u$  can be read off in a simple way from the right hand side, as explained below. And along with stronger a priori regularity of the solution, the parametrices allow increasingly weaker assumptions on the data. Moreover, the formula (1.2) clearly gives a structural information, that here is utilised to prove that additional regularity properties in subregions also carry over to the solutions.

Furthermore, as a gratis consequence of the method, the parametrix formulae may, depending on the problem and its data, yield that the solution belongs to spaces, on which the non-linear terms are of higher order than the linear terms, or are ill-defined. (Such results can often also be obtained with iteration, if the a priori information of the solution is used in each step.)

Compared to results derived from the paradifferential calculus of J.-M. Bony [Bon81], the set-up is restricted here to non-linearities of product type, as defined below, but in the present work the regularity of non-zero boundary data  $\varphi$  is taken fully into account via the term  $K\varphi$  (this was undiscussed in [Bon81]). Non-linear boundary conditions can also be covered with the present methods, but this will be a straightforward extension, and therefore left out.

As usual, the differential operator  $Au + Q(u)$  is called semi-linear when it depends linearly on the highest order derivatives of  $u$ . For such operators, it could be natural to introduce (as below) four *parameter domains*  $\mathbb{D}_\kappa$ ,  $\mathbb{D}(Q)$ ,  $\mathbb{D}(A, Q)$  and  $\mathbb{D}_u$ . Whilst the first two describe  $\{A, T\}$  ( $\kappa$  is the class of  $T$ ) and  $Q$ , the others account for spaces on which (1.1) has regularity properties resp. parametrices as expected for a semi-linear problem.

Notation and preliminaries are settled in Section 2. In a general framework the main result follows in Section 3. Some needed facts on paramultiplication are given in Section 4. In Section 5 the exact parilinearisation of non-linearities of product type is studied. Section 6 presents the consequences for the stationary von Karman problem, and the weak solutions are carried over to general  $L_p$ -Sobolev spaces. The subject of Section 7 is the parametrix and regularity results obtained for general systems of semi-linear elliptic boundary problems in vector bundles; this set-up should be natural in view of the von Karman problem treated in Section 6. Concluding remarks follow in Section 8.

**1.1. The model problem.** Throughout  $\Omega \subset \mathbb{R}^n$  is an open set with  $C^\infty$ -boundary  $\Gamma := \partial\Omega$ ;  $n \geq 2$ . It is an essential, standing assumption that  $\Omega$  is bounded. The subject is exemplified in the rest of the introduction by the following model problem, where  $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$  is the Laplacian,  $\gamma_0 u = u|_\Gamma$  the trace,

$$\begin{aligned} -\Delta u + u \cdot \partial_{x_1} u &= f \quad \text{in } \Omega, \\ \gamma_0 u &= \varphi \quad \text{on } \Gamma. \end{aligned} \tag{1.3}$$

In relation to the parametrics, (1.3) has much in common with the stationary Navier–Stokes equation, but it is not a system, so it is simpler to present.

Denoting the inverse of  $\begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix}$  by  $(R_D \ K_D)$ , where the subscript  $D$  refers to the Dirichlét problem for  $-\Delta$ , the formula (1.2) amounts to the following, when applied to a given solution  $u$ ,

$$u = P_u^{(N)}(R_D f + K_D \varphi) + (R_D L_u)^N u. \tag{1.4}$$

This expression should be new even when data and solutions are given in the Sobolev spaces  $H^s$ . But the usefulness of parametrics gets an extra dimension when the  $L_p$ -theory is discussed, so it will be natural to consider at least Sobolev spaces  $H_p^s(\overline{\Omega})$  and Hölder–Zygmund classes  $C_*^s(\overline{\Omega})$ .

However, these are special cases of Besov spaces  $B_{p,q}^s(\overline{\Omega})$  and Lizorkin–Triebel spaces  $F_{p,q}^s(\overline{\Omega})$  (the definition is recalled in (2.5)–(2.6) below), since

$$H_p^s = F_{p,2}^s \quad \text{for } 1 < p < \infty \text{ and } s \in \mathbb{R}, \tag{1.5}$$

$$C_*^s = B_{\infty,\infty}^s \quad \text{for } s \in \mathbb{R}. \tag{1.6}$$

For the well-known  $W_p^s$  spaces,  $W_p^s = B_{p,p}^s$  for non-integer  $s > 0$  and  $W_p^m = F_{p,2}^m$  for  $m \in \mathbb{N}$ ,  $1 < p < \infty$ . To avoid formulations with many scales, the exposition will be based on the  $B_{p,q}^s$  and  $F_{p,q}^s$  spaces, and for brevity  $E_{p,q}^s$  will denote a space that is either  $B_{p,q}^s$  or  $F_{p,q}^s$  (in every occurrence within, say the same formula or theorem).

Moreover,  $B_{p,q}^s(\overline{\Omega})$  and  $F_{p,q}^s(\overline{\Omega})$  are defined for  $p, q \in ]0, \infty]$  ( $p < \infty$  for  $F_{p,q}^s$ ) and  $s \in \mathbb{R}$ , where the incorporation of  $p, q < 1$  is convenient for non-linear problems, for as non-linear maps often have natural co-domains with  $p < 1$ , the  $H^s$ - and  $H_p^s$ -scales would be too tight frameworks. The price one pays for this roughly equals the burdening of the exposition that would result from a limitation to  $p, q \geq 1$ .

Furthermore,  $F_{p,1}^m$ ,  $1 \leq p < \infty$  was in [Joh04, Joh05] shown to be maximal domains for type 1, 1-operators, ie pseudo-differential operators in  $\text{OP}(S_{1,1}^m)$ ; cf Section 5.4 below. Such operators show up in the linearisations, so the  $F$ -scale is likely to appear anyway in connection with the parametrics.

If desired, the reader can of course specialise to, say  $H_p^s$  by setting  $q = 2$  in the  $F$ -scale, cf (1.5). The main part of the paper deals with the parametrix construction and its consequences, and it does not rely on a specific choice of  $L_p$ -Sobolev spaces.

For simplicity, (1.3) will in the introduction be discussed in the Besov scale  $B_{p,q}^s$ . As a basic requirement the spaces should fulfil the following two inequalities,

where for brevity  $t_+ = \max(0, t)$  stands for the positive part of  $t$ ,

$$s > \frac{1}{p} + (n-1)\left(\frac{1}{p} - 1\right)_+ \quad (1.7a)$$

$$s > \frac{1}{2} + n\left(\frac{1}{p} - \frac{1}{2}\right)_+. \quad (1.7b)$$

It is known how these allow one to make sense of the trace and the product, respectively. Working under such conditions, a main question for (1.3) is the following *inverse regularity* problem:

$$\begin{aligned} &\text{given a solution } u \text{ in one Besov space } B_{p,q}^s(\overline{\Omega}), \\ &\text{for data } f \text{ in } B_{r,o}^{t-2}(\overline{\Omega}) \text{ and } \varphi \text{ in } B_{r,o}^{t-\frac{1}{r}}(\Gamma), \\ &\text{will } u \text{ be in } B_{r,o}^t(\overline{\Omega}) \text{ too?} \end{aligned} \quad (\text{IR})$$

Consider eg a solution  $u$  in  $H^1(\overline{\Omega})$  for data  $f \in C^\alpha(\overline{\Omega})$ ,  $\varphi \in C^{2+\alpha}(\Gamma)$ ,  $0 < \alpha < 1$ . (For  $\varphi = 0$  and ‘small’  $f \in H^{-1}$  solutions exist in  $H_0^1$  for  $n = 3$  by the below Proposition 3.3.) The question is then whether  $u$  also belongs to  $C^{2+\alpha}(\overline{\Omega})$ . The latter space equals  $B_{\infty,\infty}^{2+\alpha}(\overline{\Omega})$  while  $H^1 = B_{2,2}^1$ , so problem (IR) clearly contains a classical issue; actually (IR) is somewhat sharper because of the third parameter.

In comparison with (IR), *direct* regularity properties are used for the collection of mapping properties of eg  $u \mapsto u\partial_1 u$  or  $-\Delta u + u\partial_1 u$ . An account of these clearly constitutes another regularity problem (often addressed before (IR) is solved), so it is proposed to distinguish this from (IR) by using the terms *direct/inverse*.

In connection with (IR), one purpose of this paper is to test how weak conditions one can impose in addition to (1.7). Along with this, it is described how the *parametrix* formula in (1.4) (cf also (1.19) and Theorems 3.2 and 7.6 below) yields the expected regularity properties. The result is a flexible framework implying that  $u \in B_{r,o}^t$ , also in certain cases when the map  $u \mapsto u\partial_1 u$  has higher order than  $-\Delta$  on the target space  $B_{r,o}^t$ , or when  $u\partial_1 u$  is ill-defined on  $B_{r,o}^t$ . Examples of this are given in Theorem 8.1; cf Remark 8.2.

Briefly stated, the above results and their generalisations are deduced from an exact parilinearisation  $L_u$  of  $u\partial_1 u$  together with the parametrix  $(R_D \ K_D)$  of  $\begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix}$ , belonging to the Boutet de Monvel calculus of pseudo-differential boundary operators. When combined with a Neumann series, these ingredients yield  $P_u^{(N)}$  and the parametrix formula (1.4). This resembles the usual elliptic theory at the place where non-principal terms are included, but for one thing a finite series suffices here, as in [Bon81], since the error term  $(R_D L_u)^N u$  in (1.4) only needs to belong to  $B_{r,o}^t$ ; secondly, it is less simple in the present context to keep track of the spaces on which the various steps are meaningful.

As another consequence of (1.4), if in an open subregion  $\Xi \Subset \Omega$  (ie  $\Xi$  has compact closure in  $\Omega$ , hence positive distance to the boundary) data locally have additional properties such as  $f \in B_{r_1,o_1}^{t_1-2}(\Xi, \text{loc})$ , then  $u \in B_{r_1,o_1}^{t_1}(\Xi, \text{loc})$  also holds. These local improvements are deduced from the pseudo-local property of type 1, 1-operators, which was proved recently by the author in [Joh08]; cf Section 5.4.

**1.2. On the parametrics.** It is perhaps instructive first to review the corresponding linear problem, with  $u$ ,  $f$  and  $\varphi$  as in (IR):

$$\begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix} u = \begin{pmatrix} f \\ \varphi \end{pmatrix}. \quad (1.8)$$

For the proof that  $u \in B_{r,o}^t(\overline{\Omega})$ , there is a straightforward method introduced by G. Grubb in [Gru90, Thm. 5.4] in a context of  $H_p^s$  and classical Besov spaces with  $1 < p < \infty$ .

The argument uses that  $\begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix}$  is an elliptic Green operator belonging to the calculus of L. Boutet de Monvel [BdM71], hence has a parametrix  $\begin{pmatrix} R_D & K_D \end{pmatrix}$  there (this calculus is used throughout, not just for (1.8) but also for the semi-linear problems, cf Section 7 below). As shown in [Gru90, Ex. 3.15], it is possible to take the singular Green operator part of  $R_D$  such that the class<sup>1</sup> of  $R_D$  equals

$$\text{class}(\gamma_0) - \text{order}(-\Delta) = 1 - 2 = -1. \quad (1.9)$$

With this choice,  $\begin{pmatrix} R_D & K_D \end{pmatrix}$  has continuity properties in  $H_p^s$  spaces as accounted for in [Gru90, Thm. 5.4]; under the assumptions in (1.7a), continuity from  $B_{r,o}^{t-2}(\overline{\Omega}) \oplus B_{r,o}^{t-\frac{1}{r}}(\Gamma)$  to  $B_{r,o}^t(\overline{\Omega})$  follows from [Joh96, Thm. 5.5].

Being a parametrix,  $\begin{pmatrix} R_D & K_D \end{pmatrix} \begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix} = I - \mathcal{R}$  for some regularising operator  $\mathcal{R}$  with range in  $C^\infty(\overline{\Omega})$ , and class 1 (although  $\begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix}$  is invertible,  $\mathcal{R}$  has been retained here for easier comparison with the general case). So, using the just mentioned continuity, an application of  $\begin{pmatrix} R_D & K_D \end{pmatrix}$  to both sides of (1.8) gives that

$$u = R_D f + K_D \varphi + \mathcal{R} u \quad \text{belongs to} \quad B_{r,o}^t(\overline{\Omega}). \quad (1.10)$$

This only requires the mapping properties of  $\begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix}$  and  $\begin{pmatrix} R_D & K_D \end{pmatrix}$ , that are as stated whenever  $(s, p, q)$  and  $(t, r, o)$  both satisfy (1.7a).

In the *parametrix construction* the first step is this: given a solution  $u$  of (1.3), find a *linear*,  $u$ -dependent operator  $L_u$  such that, with a sign convention,

$$L_u u = -u \partial_1 u. \quad (1.11)$$

Here it seems decisive to utilise parilinearisation. On  $\mathbb{R}^n$  this departs from paramultiplication, that yields a decomposition of the usual ‘pointwise’ product

$$v \cdot w = \pi_1(v, w) + \pi_2(v, w) + \pi_3(v, w), \quad (1.12)$$

where the  $\pi_j$  are paraproducts (cf (4.8) below). In the notation of J.-M. Bony [Bon81], paramultiplication by  $v$  is written  $T_v w$  instead of  $\pi_1(v, w)$ , and  $\pi_3(v, w) = T_w v = \pi_1(w, v)$ , whilst  $R(v, w) = vw - T_v w - T_w v = \pi_2(v, w)$  is the remainder.

More specifically, the linearisation  $L_u$  has the following form for  $\Omega = \mathbb{R}^n$ ,

$$\begin{aligned} -L_u g &= \pi_1(u, \partial_1 g) + \pi_2(u, \partial_1 g) + \pi_3(g, \partial_1 u) \\ &= T_u(\partial_1 g) + R(u, \partial_1 g) + T_{\partial_1 u}(g). \end{aligned} \quad (1.13)$$

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<sup>1</sup>The class is the minimal  $r \in \mathbb{Z} \cup \{\pm\infty\}$  with continuity  $H^r \rightarrow \mathcal{D}'$  of the operator.

Here the last line should emphasise how  $u$  and  $g$  enter. As a comparison  $g \mapsto u\partial_1 g$  can be written  $T_u(\partial_1 \cdot) + R(u, \partial_1 \cdot) + T_{\partial_1(\cdot)}(u)$ ; otherwise this notation will not be used.

In the usual parilinearisation, the  $\pi_2$ -term is omitted since it is of higher regularity (leading to the famous formula  $F(u(x)) = \pi_1(F'(u(x)), u(x)) + \text{smoother terms}$ ). But  $\pi_2(u, \partial_1 \cdot)$  is first of all *not* regularising in the present context, where  $u$  may be given in  $B_{p,q}^s$  or  $F_{p,q}^s$  also for  $s < \frac{n}{p}$  (this is possible by (1.7b)), thus allowing  $u$  to be unbounded. Secondly, (1.7b) is the only ‘non-linear’ limitation within the theory, and this arises because  $\pi_2(u, \partial_1 g)$  may or may not be defined; by incorporation of this term into  $L_u$  as in (1.13), the resulting limitation is whether or not  $L_u$  itself is defined on  $g$ .

In view of this,  $L_u$  in (1.13) is throughout referred to as an *exact* parilinearisation of  $u\partial_1 u$ . As explained in Section 5.4 below, linearisation at  $u \in B_{p_0,q_0}^{s_0}$  leads to a pseudo-differential operator in  $\text{OP}(S_{1,1}^\omega)$  for  $\omega = 1 + (\frac{n}{p_0} - s_0)_+ + \varepsilon$ . Besides the number 1 coming from  $\partial_{x_1}$ , the term  $(\frac{n}{p_0} - s_0)_+ + \varepsilon$  appears because  $u(x)$  may be unbounded on  $\Omega$  ( $\varepsilon \geq 0$ , non-trivial only for  $s_0 = \frac{n}{p_0}$ ).

As accounted for in Section 5 below,  $L_u$  has this order on *all* spaces  $B_{p,q}^s$  where it is shown to be defined; the collection of these spaces will from Section 1.3 onwards be referred to as the parameter domain of  $L_u$ , denoted by  $\mathbb{D}(L_u)$ . Moreover, the order is the same as that of  $Q(u) := u\partial_1 u$  on  $B_{p_0,q_0}^{s_0}$ . Therefore the Exact Parilinearisation Theorem (Theorem 5.7) can be summed up thus:

**Theorem 1.1.** *On every space in  $\mathbb{D}(L_u)$ , the exact parilinearisation  $g \mapsto L_u(g)$  is of the same order as the non-linear map  $Q$  on the space  $B_{p_0,q_0}^{s_0} \ni u$ .*

This is shown for arbitrary product type operators in Theorem 5.7, and in a vector bundle set-up in Theorem 7.5 below. For composition operators  $F(u(x))$  it is known that the theorem holds if  $s_0 > \frac{n}{p_0}$  since then  $L_u \in \text{OP}(S_{1,1}^0)$ .

On an open set  $\Omega \subset \mathbb{R}^n$  one can combine the linearisation in (1.13) with prolongation and restriction. When  $r_\Omega$  denotes restriction from  $\mathbb{R}^n$  to  $\Omega$ , prolongation  $\ell_\Omega$  is as usual a continuous linear map

$$\ell_\Omega: E_{p,q}^s(\overline{\Omega}) \rightarrow E_{p,q}^s(\mathbb{R}^n); \quad r_\Omega \circ \ell_\Omega = I. \quad (1.14)$$

In [Tri83, Tri92] there was given a construction, for each  $N$ , of  $\ell_\Omega$  such that (1.14) holds for  $|s| < N$ ,  $p, q > 1/N$ . While it would be possible to work with this here, it is a more convenient result of V. Rychkov [Ryc99b, Ryc99a] that  $\ell_\Omega$  can be so constructed that (1.14) holds for all  $s \in \mathbb{R}$  and all  $p, q \in ]0, \infty]$  ( $p < \infty$  in the  $F$ -case), a so-called *universal* extension operator. This construction was made for bounded Lipschitz domains. Briefly stated, the basic step is to apply a fine version of the Calderon reproducing formula  $u = \sum \varphi_v * (\psi_v * u)$  near a boundary point, where the convolution  $\psi_v * u$  (is defined when both  $u$  and  $\psi$  are supported in a cone and) has a meaningful extension by 0 to  $\mathbb{R}^n \setminus \Omega$  since it is a function; whereafter the convolution by  $\varphi_v$  gives a smooth function on  $\mathbb{R}^n$ ; the whole process is controlled in  $B_{p,q}^s$ - and  $F_{p,q}^s$ -spaces via equivalent norms involving maximal functions, established for this purpose in [Ryc99b].

Using this, the operator  $L_u$  in (1.11) is for the boundary problem (1.3) taken as

$$L_u g = -r_\Omega \pi_1(\ell_\Omega u, \partial_1 \ell_\Omega g) - r_\Omega \pi_2(\ell_\Omega u, \partial_1 \ell_\Omega g) - r_\Omega \pi_3(\ell_\Omega g, \partial_1 \ell_\Omega u). \quad (1.15)$$

As a convenient abuse, this is also called the exact parilinearisation of  $u\partial_1 u$ . It is not surprising that the mapping properties given in and before Theorem 1.1 carry over to  $L_u$  on  $\Omega$ , and this turns out to be decisive for the construction.

To focus on the simple algebra behind the parametrix formula, precise assumptions on the spaces will be suppressed until Section 1.3. First it is noted that equation (1.3), by application of  $(R_D K_D)$  and insertion of (1.11), will entail that

$$u - R_D L_u u = R_D f + K_D \phi + \mathcal{R}u. \quad (1.16)$$

The idea is now to apply the finite Neumann series

$$P_u^{(N)} := I + R_D L_u + \cdots + (R_D L_u)^{N-1}. \quad (1.17)$$

This will constitute the desired parametrix. Because  $(R_D L_u)^j$  is *linear*

$$P_u^{(N)}(I - R_D L_u) = I - (R_D L_u)^N, \quad (1.18)$$

hence the resulting parametrix formula is

$$u = P_u^{(N)}(R_D f + K_D \phi + \mathcal{R}u) + (R_D L_u)^N(u). \quad (1.19)$$

Note that in comparison with (1.10), there are two extra ingredients here, namely  $P_u^{(N)}$  and  $(R_D L_u)^N u$ , that describe the effects of the non-linear terms.

As a main application of (1.19), one can read off the regularity of a given solution  $u \in B_{p,q}^s(\overline{\Omega})$  in the following way: An uncomplicated analysis given in Theorem 3.2 below shows two new fundamental results, namely

$$\exists N: B_{p,q}^s(\overline{\Omega}) \xrightarrow{(R_D L_u)^N} B_{r,o}^t(\overline{\Omega}) \quad (1.20)$$

$$\forall N: B_{r,o}^t(\overline{\Omega}) \xrightarrow{P_u^{(N)}} B_{r,o}^t(\overline{\Omega}). \quad (1.21)$$

Since  $R_D f + K_D \phi + \mathcal{R}u$  is in  $B_{r,o}^t(\overline{\Omega})$  by the linear theory, it is therefore clear that all terms on the right hand side of (1.19) belong to  $B_{r,o}^t$ , as desired, provided  $N$  is chosen as in (1.20).

The possibility of picking  $P_u^{(N)}$  sufficiently regularising resembles the Hadamard parametrices, cf the description in [Hör85, 17.4]. It is not intended to give a symbolic calculus containing  $P_u^{(N)}$  (the difficulties in this are elucidated in Remark 5.17); it is rather a point that the parametrices and resulting regularity properties may be obtained by simpler means.

Seemingly (1.19)–(1.21) have not been crystallised before in connection with boundary problems. This might be a little surprising, since in a sense they boil down to the fact that  $R_D L_u$  is of negative order. Along with the algebra above, it is of course all-important to account for the spaces on which the various steps are both meaningful and give the conclusions (1.19)–(1.21). However, first some terminology is settled.

**1.3. Maps, orders and parameter domains.** A (possibly) non-linear operator  $T$  is said to have order  $\omega$  on  $E_{p,q}^s$  if  $T$  maps this space into  $E_{p,q}^{s-\omega}$  and  $\|T(f)|E_{p,q}^{s-\omega}\| \leq c\|f|E_{p,q}^s\|$  for some constant  $c$ . In general this leads to a function  $\omega(s, p, q)$ , for typically  $T$  is given along with a natural range of parameters  $(s, p, q)$  for which it makes sense on  $E_{p,q}^s$ ; then the set of such  $(s, p, q)$  is denoted by  $\mathbb{D}(T)$  and is called the *parameter domain* of  $T$ .

The order is differently defined if  $E_{p,q}^s$  and  $E_{p,q}^{s-\omega}$  are considered over manifolds of unequal dimensions. But here it suffices to note that for the outward normal derivative of order  $k-1$  at  $\Gamma$ , ie for  $\gamma_{k-1}f := ((\frac{\partial}{\partial \bar{n}})^{k-1}f)|_\Gamma$ , there is a well-known parameter domain  $\mathbb{D}_k$  given by

$$\mathbb{D}_k = \left\{ (s, p, q) \mid s > k + \frac{1}{p} - 1 + (n-1)\left(\frac{1}{p} - 1\right)_+ \right\}. \quad (1.22)$$

For if  $(s, p, q) \in \mathbb{D}_k$  there is continuity of the trace  $\gamma_k: B_{p,q}^s(\bar{\Omega}) \rightarrow B_{p,q}^{s-k-\frac{1}{p}}(\Gamma)$  and of  $\gamma_k: F_{p,q}^s(\bar{\Omega}) \rightarrow B_{p,p}^{s-k-\frac{1}{p}}(\Gamma)$ . The  $k^{\text{th}}$  domain  $\mathbb{D}_k$  is also the usual choice for elliptic boundary problems of class  $k \in \mathbb{Z}$ .

The notion of parameter domains (that was introduced jointly with T. Runst [JR97]) will be convenient throughout. Indeed, despite its simple nature, the model problem (1.3) requires four different parameter domains for the analysis of (IR); further below these will be introduced as  $\mathbb{D}(\mathcal{A})$ ,  $\mathbb{D}(Q)$ ,  $\mathbb{D}(\mathcal{A}, Q)$  and  $\mathbb{D}(L_u)$  along with their general analogues.

To characterise the properties leading to parametrices, let  $\mathcal{N}$  be a non-linear operator defined on  $E_{p,q}^s$  for  $(s, p, q)$  running in a parameter domain  $\mathbb{D}(\mathcal{N})$ . When compared to a linear operator  $A$  having order  $d_A$  on a domain  $\mathbb{D}(A)$ , then  $\mathcal{N}$  is said to be *A-moderate* on  $E_{p,q}^s$  in  $\mathbb{D}(A) \cap \mathbb{D}(\mathcal{N})$  if  $\mathcal{N}$  is a map  $E_{p,q}^s \rightarrow E_{p,q}^{s-\sigma}$  for some  $\sigma < d_A$ . For short  $\mathcal{N}$  is simply called *A-moderate* if such a  $\sigma$  exists on every space in  $\mathbb{D}(A) \cap \mathbb{D}(\mathcal{N})$ .

To generalise this notion, a linear operator  $L_u$  will be called a linearisation of  $\mathcal{N}$  if for every  $u \in E_{p,q}^s$  with  $(s, p, q)$  in  $\mathbb{D}(\mathcal{N})$ ,

$$\mathcal{N}(u) = -L_u(u). \quad (1.23)$$

Here  $L_u$  should be a meaningful linear operator parametrised by the  $u$  (running through the spaces) in  $\mathbb{D}(\mathcal{N})$ , or possibly for  $u$  in a larger parameter domain  $\mathbb{D}(\mathcal{L})$ .

It will be required that, for  $u \in E_{p_0,q_0}^{s_0}$  fixed,  $g \mapsto L_u(g)$  should be of order  $\omega(s, p, q)$ , ie be a map  $E_{p,q}^s \rightarrow E_{p,q}^{s-\omega(s,p,q)}$ , on every  $E_{p,q}^s$  in a parameter domain denoted  $\mathbb{D}(L_u)$ . (It will be seen in Theorem 5.7, ie the full version of the Exact Paralinearisation Theorem, that  $\mathbb{D}(\mathcal{L}) = \mathbb{R} \times ]0, \infty]^2$  because the operator  $L_u$  is a meaningful object for all  $u$ ; but once  $u$  is fixed, the parameter domain of  $g \mapsto L_u(g)$  is much smaller, and its determination is a main point in Theorem 5.7.) Although  $\omega$  is a function  $\omega(s, p, q, s_0, p_0, q_0)$ , the arguments  $s_0, p_0, q_0$  are often left out, since  $u$  is fixed in  $E_{p_0,q_0}^{s_0}$ ; but for generality's sake  $(s, p, q)$  is kept though  $\omega$  often is a constant in this paper.



**Definition 1.2.** A linearisation  $L_u$  with parameter domain  $\mathbb{D}(L_u) \supset \mathbb{D}(\mathcal{N})$  is said to be *moderate* if, for every linearisation point  $u$  in an arbitrary  $E_{p_0, q_0}^{s_0}$  in  $\mathbb{D}(\mathcal{N})$ ,

$$\omega_{\max} := \sup_{\mathbb{D}(L_u) \times \mathbb{D}(\mathcal{N})} \omega(s, p, q, s_0, p_0, q_0) < \infty. \quad (1.24)$$

In case there is some  $(s_0, p_0, q_0)$  in  $\mathbb{D}(\mathcal{N})$  such that  $\sup_{(s, p, q) \in \mathbb{D}(L_u)} \omega(s, p, q) < \infty$ , then  $L_u$  is said to be *moderate on  $E_{p_0, q_0}^{s_0}$* . And  $L_u$  is said to be *A-moderate on  $E_{p_0, q_0}^{s_0} \ni u$*  if  $(s, p, q) \in \mathbb{D}(A) \cap \mathbb{D}(L_u)$  implies

$$\omega(s, p, q, s_0, p_0, q_0) < d_A. \quad (1.25)$$

Moderate linearisations are therefore those that, regardless of the linearisation point  $u$ , have uniformly bounded orders on their entire parameter domains. Clearly  $\mathcal{N}$  is A-moderate on  $E_{p_0, q_0}^{s_0}$  (in  $\mathbb{D}(\mathcal{N})$ ) if  $L_u$  is so, for since  $-L_u u = \mathcal{N}(u)$  holds at  $(s_0, p_0, q_0)$  it is trivial that  $\mathcal{N}$  is a map  $E_{p_0, q_0}^{s_0} \rightarrow E_{p_0, q_0}^{s_0 - \omega(s_0, p_0, q_0)} \subset E_{p_0, q_0}^{s_0 - d_A}$ .

*Remark 1.3.* With the third term of (1.11) equal to  $r_{\Omega} \pi_3(\ell_{\Omega} \cdot, \partial_1 \ell_{\Omega} u)$ , the regularity of  $L_u g$  is known to depend mainly on  $g$ . Indeed, if  $u \in B_{p_0, q_0}^{s_0}(\overline{\Omega})$ , then  $L_u g$  has in general only  $(\frac{n}{p_0} - s_0)_+ + 1 + \varepsilon$  derivatives less than  $g$ ; cf Theorem 1.1. This value is a constant independent of  $g$  and  $\frac{n}{p_0} - s_0 < \frac{n}{2}$  holds by (1.7b), so  $\omega_{\max} < \infty$  and  $L_u$  is moderate; and  $\Delta$ -moderate if eg  $s_0 \geq \frac{n}{p_0}$ .

The linearisation  $g \mapsto u \partial_1 g$  might look natural, but since  $u \partial_1 g \in B_{p, q}^s$  can be shown to hold if  $s \leq s_0$ , it is of non-constant order  $\omega(t, r, o) \geq t - s_0$  on  $B_{r, o}^t \ni g$ , hence not moderate because  $\omega_{\max} \geq \sup_t t - s_0 = \infty$ . Moreover, this order is larger than that of  $-\Delta$  when  $t > s_0 + 2$ , so in this region it is not  $\Delta$ -moderate.

Before justifying the formal steps in (1.16)–(1.19), it is convenient to present the parameter domains for problem (IR) first. This is done by merely stating the consequences of the following sections, with reference to the general results.

Departing from the linear part of (1.3), the Dirichlét condition leads to (1.7a), and since the problem has class 1, one can reformulate this using (1.22), by introduction of the parameter domain of  $\mathcal{A} = (\frac{-\Delta}{\gamma_0})$  as

$$\mathbb{D}(\mathcal{A}) = \mathbb{D}_1 = \{(s, p, q) \mid s > \frac{1}{p} + (n-1)(\frac{1}{p} - 1)_+\}. \quad (1.26)$$

For the quadratic operator  $Q(u) := u \partial_1 u$  one should have a parameter domain  $\mathbb{D}(Q)$  such that  $Q$  is well defined on all  $B_{p, q}^s$  and  $F_{p, q}^s$  in this domain. This question is treated in Proposition 5.5 below, in a context of product type operators studied in Section 5.1. This yields precisely the condition (1.7b), cf (5.9) and Figure 2 there; this amounts to the *quadratic* standard domain of  $Q$ ,

$$\mathbb{D}(Q) = \{(s, p, q) \mid s > \frac{1}{2} + (\frac{n}{p} - \frac{n}{2})_+\}. \quad (1.27)$$

In the important determination of the spaces on which  $Q$  is  $\Delta$ -moderate, one can depart from the conclusion of Proposition 5.5 below that  $Q$  is of order  $\sigma(s, p, q) = 1 + (\frac{n}{p} - s)_+ + \varepsilon$ , with an  $\varepsilon \geq 0$  nontrivial only for  $s = \frac{n}{p}$ . Ie  $Q$  is a bounded map

$$Q: B_{p, q}^s \rightarrow B_{p, q}^{s - \sigma(s, p, q)}. \quad (1.28)$$

(More precisely, one should instead of  $Q$  consider  $\begin{pmatrix} Q \\ 0 \end{pmatrix}$  and check where it is  $\mathcal{A}$ -moderate, but it is a convenient abuse to focus on  $Q$  and  $\Delta$  instead.)

In principle one can now introduce a parameter domain of  $\Delta$ -moderacy for  $Q$  by solving the inequality  $\sigma(s, p, q) < 2$  on  $\mathbb{D}(\mathcal{A}) \cap \mathbb{D}(Q)$ , cf (1.25); this leads to

$$\mathbb{D}(\mathcal{A}, Q) := \mathbb{D}(\mathcal{A}) \cap \{ (s, p, q) \in \mathbb{D}(Q) \mid \sigma(s, p, q) < 2 \}. \quad (1.29)$$

However, this calculation is made for a general semi-linear problem with the result summed up in Corollary 5.9 below. If  $n \geq 3$  for simplicity, one finds from this result and the obvious inclusion  $\mathbb{D}(\mathcal{A}) = \mathbb{D}_1 \subset \mathbb{D}(Q)$  that

$$\mathbb{D}(\mathcal{A}, Q) = \{ (s, p, q) \mid s > \frac{1}{2} + (\frac{n}{p} - \frac{3}{2})_+ \}. \quad (n \geq 3) \quad (1.30)$$

So far the considerations are classical in nature (even if formulated for the  $B_{p,q}^s$ -spaces). But the use of parameter domains and the concise  $\mathbb{D}$ -notation will be particularly useful for the next remarks, that also explain how general regularity results the present methods can give.

Using the exact parilinearisation,  $Q(u) = -L_u(u)$  holds on the entire quadratic standard domain  $\mathbb{D}(Q)$ , as verified in Lemma 5.4 below. But as a new observation,  $g \mapsto L_u(g)$  is for a fixed  $u \in B_{p_0, q_0}^{s_0}$  defined on every space in

$$\mathbb{D}(L_u) = \{ (s, p, q) \mid s > 1 - s_0 + (\frac{n}{p} + \frac{n}{p_0} - n)_+ \}. \quad (1.31)$$

This is part of the content of the Exact Parilinearisation Theorem in Section 5.2 below.

It is not difficult to infer that  $\mathbb{D}(L_u) \supset \mathbb{D}(Q)$  holds for  $(s_0, p_0, q_0) \in \mathbb{D}(Q)$ , in general with a considerable gap—for the borderline of  $\mathbb{D}(Q)$  is obtained from  $\mathbb{D}(L_u)$  by setting  $(s, p, q)$  and  $(s_0, p_0, q_0)$  equal, so when  $(s_0, p_0, q_0) \in \mathbb{D}(Q)$ , then  $(s, p, q)$  can lie an exterior part of  $\mathbb{D}(Q)$  without violating the inequality in (1.31). It is also clear that  $\mathbb{D}(L_u)$  increases with improving a priori regularity of  $u$ , ie with increasing  $s_0$  or  $p_0$ .

Moreover, given a solution  $u$  in some  $B_{p_0, q_0}^{s_0}$  in  $\mathbb{D}(\mathcal{A}, Q)$ , the parametres and the resulting inverse regularity properties are established in the domain

$$\mathbb{D}_u = \mathbb{D}_1 \cap \mathbb{D}(L_u). \quad (1.32)$$

This is larger than  $\mathbb{D}(\mathcal{A}, Q)$ , for (1.29) gives  $\mathbb{D}(\mathcal{A}, Q) \subset \mathbb{D}_1 \cap \mathbb{D}(Q) \subset \mathbb{D}_1 \cap \mathbb{D}(L_u)$ .

It is now possible to sketch a proof of the parametrix formula (1.19) and the crucial properties in (1.21)–(1.20). Given a solution  $u$  of (1.3) in, say  $B_{p_0, q_0}^{s_0}$  with  $(s_0, p_0, q_0)$  in  $\mathbb{D}(\mathcal{A}, Q)$ , Theorem 1.1 shows that  $L_u$  has order  $\sigma(s_0, p_0, q_0) < 2$  on all spaces in  $\mathbb{D}(L_u)$ . Therefore  $R_D L_u$  is defined and has order  $-\delta$ , for some  $\delta > 0$ , on all spaces  $B_{p,q}^s$  in  $\mathbb{D}_u$ . Since  $\mathbb{D}_u$  is upwards unbounded, the composite  $(R_D L_u)^N$  is defined and has order  $-N\delta$  on  $\mathbb{D}_u$ . So via embeddings,  $(R_D L_u)^N$  maps any  $B_{p,q}^s$  in  $\mathbb{D}_u$  to  $C^k(\overline{\Omega})$  for all sufficiently large  $N$ , hence it fulfills (1.20). (This breaks down for the other linearisations in Remark 1.3, since they are not moderate.) Clearly  $(R_D L_u)^N$  is then also of order 0 on every space in  $\mathbb{D}_u$ , so since  $P_u^{(N)}$  is a sum of such powers, it satisfies (1.21). Then (1.16)–(1.19) follow as identities in  $B_{p_0, q_0}^{s_0}$ , since this space is in  $\mathbb{D}(\mathcal{A}, Q) \subset \mathbb{D}_u$ , in particular the parametrix formula is obtained. As seen after (1.21) this also gives the desired regularity  $u \in B_{r,o}^t$  at once.

The deduction of the parametrix formula after (1.32) is of course rather straightforward. However, this is partly because a few commutative diagrams have been suppressed in the explanation. Moreover, it is easy to envisage that the arguments extend to a whole range of eg semi-linear elliptic problems, and perhaps it is most natural to comment on the generalisations first.

For other problems the domain  $\mathbb{D}(\mathcal{A}, Q)$  of  $\mathcal{A}$ -moderacy will generally be more complicated than the polygon in (1.30). Eg it may be non-convex and operators corresponding to  $(R_D L_u)^N$  can have orders bounded with respect to  $N$  (unlike  $-N\delta$ ). This is the case for composition type problems with  $Q(u) = F \circ u$  in [JR97]; cf Figure 1 there. Furthermore, the parameter domains can be ‘tight’ in the sense that they (unlike the above examples) need not be upwards unbounded; here parabolic initial and boundary value problems could be mentioned, for if the given data only fulfill finitely many compatibility conditions, then solutions can only exist in  $B_{p,q}^s$  for  $s$  below a certain limit. Cf [Gru95] for the determination of specific compatibility conditions for fully inhomogeneous problems.

In view of this, it seems practical to assume only that the parameter domain  $\mathbb{D}_u$  is connected. Under this hypothesis it is possible to prove the existence of the desired  $N$  in (1.20) by continuous induction along an arbitrary curve from  $(\frac{n}{p}, s)$  to  $(\frac{n}{r}, t)$ , running inside the parameter domain  $\mathbb{D}_u$ .

These techniques are presented in Section 3, where the brief argument after (1.32) is replaced by an analytical proof of the parametrix formula. In fact the set-up in Section 3 is both axiomatic and general, allowing also parabolic problems and linearisations of non-constant order. The last aspect might be important for problems with linearisations of  $F(u(x))$  at unbounded solutions  $u$ .

However, this paper mainly focuses on non-linearities with a product structure, as there are ample examples of such problems, and because more general classes would burden the exposition with more technicalities, or even atypical phenomena. Therefore generalised multiplication is reviewed in Section 4, and a class of non-linearities of product type has been introduced in Section 5; these are of the form  $P_2(P_0 u \cdot P_1 u)$  for linear differential operators  $P_j$  with constant coefficients. As an example the von Karman equation is treated in Section 6. The abstract results of Section 3 are exploited systematically in Section 7 on general systems of semi-linear elliptic boundary problems of product type.

## 2. PRELIMINARIES

**2.1. Notation.** For simplicity  $t_{\pm} := \max(0, \pm t)$  for  $t \in \mathbb{R}$ . The bracket  $\llbracket A \rrbracket$  stands for 1 and 0 when the assertion  $A$  is true resp. false. When  $\alpha \in \mathbb{N}_0^n$  is a multiindex,  $D^{\alpha} := (-i)^{|\alpha|} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$  where  $|\alpha| = \alpha_1 + \dots + \alpha_n$ .

The space of smooth functions with compact support is denoted by  $C_0^{\infty}(\Omega)$  or  $\mathcal{D}(\Omega)$ , when  $\Omega \subset \mathbb{R}^n$  is open;  $\mathcal{D}'(\Omega)$  is the dual space of distributions on  $\Omega$ .  $\langle u, \varphi \rangle$  denotes the action of  $u \in \mathcal{D}'(\Omega)$  on  $\varphi \in C_0^{\infty}(\Omega)$ . The restriction  $r_{\Omega}: \mathcal{D}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\Omega)$  is the transpose of the extension by 0 outside of  $\Omega$ , denoted  $e_{\Omega}: C_0^{\infty}(\Omega) \rightarrow C_0^{\infty}(\mathbb{R}^n)$ . Using this,  $C^{\infty}(\overline{\Omega}) = r_{\Omega} C^{\infty}(\mathbb{R}^n)$  etc.

The Schwartz space of rapidly decreasing  $C^\infty$ -functions is written  $\mathcal{S}$  or  $\mathcal{S}(\mathbb{R}^n)$ , while  $\mathcal{S}'(\mathbb{R}^n)$  stands for the space of tempered distributions. The Fourier transformation of  $u$  is  $\mathcal{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) dx$ , with inverse  $\mathcal{F}^{-1}v(x) = \check{v}(x)$ . The space of slowly increasing functions, ie  $C^\infty$ -functions  $f$  fulfilling  $|D^\alpha f(x)| \leq c_\alpha \langle x \rangle^{N_\alpha}$  for all multindices  $\alpha$  is written  $\mathcal{O}_M(\mathbb{R}^n)$ ; hereby  $\langle x \rangle = (1 + |x|^2)^{1/2}$ .

The singular support of  $u \in \mathcal{D}'$ , denoted  $\text{sing supp } u$ , is the complement of the largest open set on which  $u$  acts a  $C^\infty$ -function. Outside of  $F := \text{sing supp } u$ , mollification behaves as nicely as one could expect (the following could be folklore): for  $\psi \in C_0^\infty(\mathbb{R}^n)$ , with  $\psi_k(x) = \varepsilon_k^{-n} \psi(\varepsilon_k^{-1}x)$  for  $0 \leq \varepsilon_k \rightarrow 0$ , one has

$$\psi_k * u \rightarrow c_0 u \quad \text{in } C^\infty(\mathbb{R}^n \setminus F); \quad c_0 = \int \psi dx. \quad (2.1)$$

For if  $K \Subset \mathbb{R}^n$  with  $K \cap F = \emptyset$  and  $1 = \varphi + \eta$  with  $\varphi \in C_0^\infty(\mathbb{R}^n)$  and  $\text{supp } \varphi \cap F = \emptyset$ ,  $K \cap \text{supp } \eta = \emptyset$ , uniform continuity of  $D^\alpha(\varphi u)$  gives  $\sup_K |D^\alpha(\psi_k * \varphi u - c_0 \varphi u)| \searrow 0$ . And by the theorem of supports,  $\psi_k * (\eta u) = 0$  near  $K$ , eventually.

It will later be convenient that this holds more generally for  $\psi \in \mathcal{S}(\mathbb{R}^n)$ , even though the convolution  $\psi_k * (\eta u)$  need not vanish in  $K$ :

**Lemma 2.1.** *For  $u \in \mathcal{S}'(\mathbb{R}^n)$ ,  $\psi \in \mathcal{S}(\mathbb{R}^n)$ , the regularising sequence  $\psi_k * u$  converges to  $(\int \psi dx) \cdot u$  in the  $C^\infty$ -topology on  $\mathbb{R}^n \setminus \text{sing supp } u$ ; ie it fulfills (2.1).*

*Proof.* Continuing the above, one has  $0 < \text{dist}(K, \text{supp } \eta) \leq \varepsilon_k \langle \varepsilon_k^{-1}(x - y) \rangle$  for  $x \in K$ ,  $y \in \text{supp } \eta$ , and

$$|\langle \eta u, D_x^\alpha \psi_k(x - \cdot) \rangle| \leq c \sup_{y \in \mathbb{R}^n, |\beta| \leq N} \langle y \rangle^N \left| D_y^\beta \left( \frac{\eta(y)}{\varepsilon_k^{n+|\alpha|}} D^\alpha \psi \left( \frac{x-y}{\varepsilon_k} \right) \right) \right|. \quad (2.2)$$

Since  $\langle y \rangle^N \leq c_K \langle \frac{x-y}{\varepsilon_k} \rangle^N$  for  $\varepsilon_k < 1$ , and powers of  $\langle \varepsilon_k^{-1}(x - y) \rangle$  may be absorbed in an  $\mathcal{S}$ -seminorm on  $\psi$ , it follows that  $\sup_K |D^\alpha(\psi_k * (\eta u))| \leq C \varepsilon_k \searrow 0$ .  $\square$

*Remark 2.2.* Lemma 2.1 was called the Regular Convergence Lemma (and  $\mathbb{R}^n \setminus \text{sing supp } u$  the regular set of  $u$ ) in [Joh08], where it played a significant role in investigations of type 1, 1-operators and products.

**2.2. Spaces.** Norms and quasi-norms are written  $\|x|X\|$  for  $x$  in a vector space  $X$ ; recall that  $X$  is quasi-normed if the triangle inequality is replaced by the existence of  $c \geq 1$  such that all  $x$  and  $y$  in  $X$  fulfil  $\|x+y|X\| \leq c(\|x|X\| + \|y|X\|)$  (“quasi-” will be suppressed when the meaning is settled by the context). Eg  $L_p(\mathbb{R}^n)$  and  $\ell_p(\mathbb{N})$  for  $p \in ]0, \infty]$  are quasi-normed with  $c = 2^{(\frac{1}{p}-1)_+}$ ; this is seen because both  $\ell_p$  and  $L_p$  for  $0 < p \leq 1$  satisfy the following, for  $\lambda = p$ ,

$$\|f + g\| \leq (\|f\|^\lambda + \|g\|^\lambda)^{1/\lambda}, \quad (2.3)$$

where on the right Hölder’s inequality applies to the dual exponents  $1/p$  and  $1/(1-p)$ .

For brevity  $\|f\|_p := \|f|L_p\|$  for  $f \in L_p(\Omega)$ , with  $\Omega \subset \mathbb{R}^n$  an open set.  $X_1 \oplus X_2$  denotes the product space topologised by  $\|x_1|X_1\| + \|x_2|X_2\|$ . For a bilinear operator  $B: X_1 \oplus X_2 \rightarrow Y$ , continuity is equivalent to boundedness and to existence

of a constant  $c$  such that  $\|B(x_1, x_2)|Y\| \leq c\|x_1\|X_1\|\|x_2\|X_2\|$ . In the affirmative case, the least possible  $c$  is the operator norm  $\|B\| = \sup\{\|B(x_1, x_2)|Y\| \mid \text{for } j = 1, 2: \|x_j\|X_j\| \leq 1\}$ .

The spaces  $B_{p,q}^s(\mathbb{R}^n)$  and  $F_{p,q}^s(\mathbb{R}^n)$  are, with conventions as in [Yam86a], defined as follows: First a Littlewood–Paley decomposition is constructed using a function  $\Psi$  in  $C^\infty(\mathbb{R})$  for which  $\Psi \equiv 0$  and  $\Psi \equiv 1$  holds for  $t \geq 13/10$  and  $t \leq 11/10$ , respectively. Then  $\Psi_j(\xi) := \Psi(2^{-j}|\xi|)$  and

$$\Phi_j(\xi) = \Psi_j(\xi) - \Psi_{j-1}(\xi) \quad (\Psi_{-1} \equiv 0) \quad (2.4)$$

gives  $\Psi_j = \Phi_0 + \dots + \Phi_j$  for every  $j \in \mathbb{N}_0$ , hence  $1 \equiv \sum_{j=0}^\infty \Phi_j$  on  $\mathbb{R}^n$ . As a shorthand  $\varphi(D)$  will denote the pseudo-differential operator with symbol  $\varphi$ , ie  $\varphi(D)u = \mathcal{F}^{-1}(\varphi \cdot \mathcal{F}u)$ , say for  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ .

For a *smoothness index*  $s \in \mathbb{R}$ , an *integral-exponent*  $p \in ]0, \infty]$  and *sum-exponent*  $q \in ]0, \infty]$ , the *Besov space*  $B_{p,q}^s(\mathbb{R}^n)$  and the *Lizorkin–Triebel space*  $F_{p,q}^s(\mathbb{R}^n)$  are defined as

$$B_{p,q}^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \|\{2^{sj} \|\Phi_j(D)u(\cdot)\|_{L_p}\}_{j=0}^\infty\|_{\ell_q} < \infty\}, \quad (2.5)$$

$$F_{p,q}^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \|\{2^{sj} \Phi_j(D)u\}_{j=0}^\infty\|_{\ell_q(\cdot)}\|_{L_p} < \infty\}. \quad (2.6)$$

Throughout it will be tacitly understood that  $p < \infty$  whenever Lizorkin–Triebel spaces are under consideration.  $B_{p,q}^s(\mathbb{R}^n; \text{loc})$  etc. denote the spaces of distributions that locally belong to the above ones.

The spaces are described in eg [RS96, Tri83, Tri92, Yam86a]. They are quasi-Banach spaces with the quasi-norms given by the finite expressions in (2.5) and (2.6). Using (2.3) twice, they are seen to fulfill (2.3) for  $\lambda = \min(1, p, q)$ .

Among the embedding properties of these spaces one has  $B_{p,q}^s \hookrightarrow B_{p,q}^{s-\varepsilon}$  for  $\varepsilon > 0$ , and if in the second line  $\Omega \subset \mathbb{R}^n$  is open and bounded with  $B_{p,q}^s(\overline{\Omega}) := r_\Omega B_{p,q}^s(\mathbb{R}^n)$  endowed with the infimum norm,

$$B_{p,q}^s \hookrightarrow B_{r,o}^t \quad \text{for} \quad s - \frac{n}{p} = t - \frac{n}{r}, \quad p > r; \quad o = q, \quad (2.7)$$

$$B_{p,q}^s(\overline{\Omega}) \hookrightarrow B_{r,q}^s(\overline{\Omega}) \quad \text{for} \quad p \geq r. \quad (2.8)$$

The analogous holds for  $F_{p,q}^s$ , except that  $F_{p,q}^s \hookrightarrow F_{r,o}^t$  if only  $s - \frac{n}{p} = t - \frac{n}{r}$ ,  $p > r$ . Moreover,  $B_{p,q}^s \hookrightarrow L_\infty$  holds if and only if  $s > n/p$  or both  $s = n/p$  and  $q \leq 1$ ; and  $F_{p,q}^s \hookrightarrow L_\infty$  if and only if  $s > n/p$  or both  $s = n/p$  and  $p \leq 1$ .

For the reader's sake a few lemmas are recalled. They are concerned with convergence of a series  $\sum_{j=0}^\infty u_j$  fulfilling the dyadic ball *condition*: for some  $A > 0$

$$\text{supp } \mathcal{F}u_j \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq A2^j\}, \quad \text{for } j \geq 0. \quad (2.9)$$

**Lemma 2.3** (The dyadic ball criterion). *Let  $s > \max(0, \frac{n}{p} - n)$  for  $p, q \in ]0, \infty]$  and suppose  $u_j \in \mathcal{S}'(\mathbb{R}^n)$  fulfil (2.9) and*

$$B := \left(\sum_{j=0}^\infty 2^{sjq} \|u_j\|_p^q\right)^{\frac{1}{q}} < \infty. \quad (2.10)$$

Then  $\sum_{j=0}^{\infty} u_j$  converges in  $\mathcal{S}'(\mathbb{R}^n)$  to some  $u$  lying in  $B_{p,q}^s(\mathbb{R}^n)$  and  $\|u\|_{B_{p,q}^s} \leq cB$  for some  $c > 0$  depending on  $n, s, p$  and  $q$ .

**Lemma 2.4** (The dyadic ball criterion). *Let  $s > \max(0, \frac{n}{p} - n)$  for  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and suppose  $u_j \in \mathcal{S}'(\mathbb{R}^n)$  fulfil (2.9) and*

$$F(q) := \left\| \left( \sum_{j=0}^{\infty} 2^{sjq} |u_j(\cdot)|^q \right)^{\frac{1}{q}} \right\|_p < \infty. \quad (2.11)$$

Then  $\sum_{j=0}^{\infty} u_j$  converges in  $\mathcal{S}'(\mathbb{R}^n)$  to some  $u$  lying in  $F_{p,r}^s(\mathbb{R}^n)$  for

$$r \geq q, \quad r > \frac{n}{n+s}, \quad (2.12)$$

and  $\|u\|_{F_{p,r}^s} \leq cF(r)$  for some  $c > 0$  depending on  $n, s, p$  and  $r$ .

This follows from the usual version in which  $s > \max(0, \frac{n}{p} - n, \frac{n}{q} - n)$  is required, for one can just pass to larger values of  $q$  if necessary. Lemma 2.4 emphasises that the interrelationship between  $s$  and  $q$  is inconsequential for the mere existence of the sum.

It is also well known that the restrictions on  $s$  can be entirely removed if  $\sum u_j$  fulfils the dyadic *corona* condition: for some  $A > 0$ ,  $\text{supp } \mathcal{F}u_0 \subset \{|\xi| \leq A\}$  and

$$\text{supp } \mathcal{F}u_j \subset \{\xi \in \mathbb{R}^n \mid \frac{1}{A}2^j \leq |\xi| \leq A2^j\}, \quad \text{for } j > 0. \quad (2.13)$$

**Lemma 2.5** (The dyadic corona criterion). *Let  $u_j \in \mathcal{S}'(\mathbb{R}^n)$  fulfil (2.13) and (2.10). Then  $\sum_{j=0}^{\infty} u_j$  converges in  $\mathcal{S}'(\mathbb{R}^n)$  to some  $u$  for which  $\|u\|_{B_{p,q}^s} \leq cB$  for some  $c > 0$  that depends on  $n, s, p$  and  $q$ . And similarly for  $F_{p,q}^s(\mathbb{R}^n)$ , if  $F(q) < \infty$ .*

These lemmas are proved in eg [Yam86a]. To estimate the numbers  $B$  and  $F$  in the above criteria, the following summation lemma is often useful: for any sequence  $(a_j)$  in  $\mathbb{C}$ ,  $s < 0$  and  $q, r \in ]0, \infty]$ ,

$$\left( \sum_{j=0}^{\infty} 2^{sjq} \left( \sum_{k=0}^j |a_k|^r \right)^{\frac{q}{r}} \right)^{\frac{1}{q}} \leq c(s, q, r) \|2^{sj} a_j\|_{\ell_q}. \quad (2.14)$$

For  $s > 0$  the analogous holds if the second sum is over  $k \geq j$  instead. (Cf [Yam86a, Lem. 3.8] for  $r = 1$ .)

For the estimates of the exact parilinearisation in Section 5.3 and 5.4, the following vector-valued Nikol'skiĭ–Plancherel–Polya inequality will be convenient.

**Lemma 2.6.** *Let  $0 < r < p < \infty$ ,  $0 < q \leq \infty$  and  $A > 0$ . There is a constant  $c$  such that for every sequence of functions  $f_k \in L_r(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$  with  $\text{supp } \mathcal{F}f_k \subset B(0, A2^k)$ ,*

$$\left\| \left( \sum_{k=0}^{\infty} |f_k|^q \right)^{1/q} \right\|_{L_p} \leq c \left\| \sup_k 2^{(\frac{n}{r} - \frac{n}{p})k} |f_k| \right\|_{L_r}. \quad (2.15)$$

The ordinary Nikol'skiĭ–Plancherel–Polya inequality results from this if  $f_k \neq 0$  holds only for one value of  $k$ . (Lemma 2.6 itself can be reduced to this version by means of an elementary inequality in [BM01, Lem. 4], cf [JS07].)

To treat the examples in Proposition 2.10 below tensor products will be useful. However, lacking a thorough reference to this, the next result is given. It improves

[JR97, Prop. 2.7] by including the case  $s > 0$ . A proof using the above dyadic corona criterion is supplied, partly because [JR97, Prop. 2.7] was stated without details, partly because it then is more natural to omit the details behind the better known, but analogous, paraproduct estimates recalled in Remark 4.3 below.

**Lemma 2.7.** *The continuous map  $(u, v) \mapsto u \otimes v$  from  $\mathcal{S}'(\mathbb{R}^{n'}) \times \mathcal{S}'(\mathbb{R}^{n''})$  to  $\mathcal{S}'(\mathbb{R}^{n'+n''})$  restricts to bounded bilinear maps*

$$B_{p,q}^s(\mathbb{R}^{n'}) \times B_{p,q}^s(\mathbb{R}^{n''}) \rightarrow B_{p,q}^s(\mathbb{R}^{n'+n''}) \quad \text{for } s > 0, \quad (2.16)$$

$$B_{p,q}^{s'}(\mathbb{R}^{n'}) \times L_p(\mathbb{R}^{n''}) \rightarrow B_{p,q}^{s'}(\mathbb{R}^{n'+n''}) \quad \text{for } s' < 0, 1 \leq p \leq \infty, \quad (2.17)$$

$$B_{p,q'}^{s'}(\mathbb{R}^{n'}) \times B_{p,q''}^{s''}(\mathbb{R}^{n''}) \rightarrow B_{p,q}^{s'+s''}(\mathbb{R}^{n'+n''}) \quad \text{for } s', s'' < 0, \frac{1}{q} = \frac{1}{q'} + \frac{1}{q''}. \quad (2.18)$$

*Proof.* For  $u \in \mathcal{S}'(\mathbb{R}^{n'})$  and  $v \in \mathcal{S}'(\mathbb{R}^{n''})$  there is a decomposition, when  $\Psi'_N = \Phi'_0 + \dots + \Phi'_N$  refers to a Littlewood–Paley decomposition on  $\mathbb{R}^{n'}$  with the present conventions, so that  $u_k = \Phi'_k(D)u$ ,  $u^k = \Psi'_k(D)u$ , and similarly for  $v$  on  $\mathbb{R}^{n''}$ ,

$$u \otimes v = \lim_{N \rightarrow \infty} \mathcal{F}^{-1}((\Psi'_N \otimes \Psi''_N) \mathcal{F}(u \otimes v)) = \sum_{k=0}^{\infty} (u_k v^{k-1} + u^k v_k). \quad (2.19)$$

Both series on the right-hand side fulfill the dyadic corona condition (2.13), since  $\xi = (\xi', \xi'')$  for  $\xi' \in \mathbb{R}^{n'}$ ,  $\xi'' \in \mathbb{R}^{n''}$  and  $|(\xi', 0)| \leq |\xi| \leq |\xi'| + |\xi''|$  yield eg

$$\xi \in \text{supp } \mathcal{F}(u_k v^{k-1}) \implies \frac{11}{20} 2^k \leq |\xi| \leq \frac{13}{10} (2^k + 2^{k-1}) = \frac{39}{20} 2^k. \quad (2.20)$$

For  $1 \leq p \leq \infty$  the usual convolution estimate gives

$$2^{sk} \|u_k v^{k-1}\|_{L_p(\mathbb{R}^{n'+n''})} \leq 2^{ks} \|u_k\|_p \|\mathcal{F}^{-1} \Psi''\|_1 \|v\|_p, \quad (2.21)$$

and since  $B_{p,q}^s \hookrightarrow L_p$  for  $s > 0$ ,  $p \geq 1$ , it follows from Lemma 2.5 by calculation of the  $\ell_q$ -norms that

$$\left\| \sum u_k v^{k-1} \right\|_{B_{p,q}^s(\mathbb{R}^{n'+n''})} \leq c \|u\|_{B_{p,q}^s(\mathbb{R}^{n'})} \|v\|_{B_{p,q}^s(\mathbb{R}^{n''})}. \quad (2.22)$$

The other series is treated the same way, and thus follows (2.16) for  $p \geq 1$ . For  $p < 1$  one has  $\|v^{k-1}\|_p \leq \|v_0\| + \dots + \|v_k\|_p \leq \|v\|_{F_{p,1}^0} \leq c \|v\|_{B_{p,q}^s}$ , and this instead of (2.21) extends (2.16) to all  $p \in ]0, \infty]$ .

Since (2.21) holds for all  $s$ , it suffices for (2.17) to estimate  $\sum u^k v_k$ . By (2.14),

$$\sum 2^{s'kq} \|u^k\|_p^q \leq \sum 2^{s'kq} (\|u_0\|_p + \dots + \|u_k\|_p)^q \leq c \sum 2^{s'kq} \|u_k\|_p^q. \quad (2.23)$$

Using Lemma 2.5, it follows as above that  $\|\sum u^k v_k\|_{B_{p,q}^{s'}} \leq c \|u\|_{B_{p,q}^{s'}} \|v\|_p$ .

To prove (2.18) one can use the summation lemma for both  $u^k$  and  $v^{k-1}$  since both  $s'$ ,  $s'' < 0$ . Combining this with Hölder's inequality for  $\ell_q$ , the above procedure gives a bound of  $\|u \otimes v\|_{B_{p,q}^{s'+s''}}$  by  $c \|u\|_{B_{p,q'}^{s'}} \|v\|_{B_{p,q''}^{s''}}$ .  $\square$

**2.3. Examples.** The delta measure  $\delta_0 \in B_{p,\infty}^{\frac{n}{p}-n}(\mathbb{R}^n)$  for  $0 < p \leq \infty$ ; this well-known fact follows directly from (2.5) since  $2^{j(\frac{n}{p}-n)}\|\check{\Phi}_j\|_p$  is  $j$ -independent.

Other examples include  $|x|^a$ , that for  $a > -n$  and  $p \geq 2$  was shown in [Yam88] to be locally in  $B_{p,\infty}^{\frac{n}{p}+a}(\mathbb{R}^n)$  at  $x = 0$ . For  $0 < p \leq \infty$  there is a technical treatment via differences in [RS96, Sect.2.3] of  $|x|^a |\log |x||^{-b}$ , but without details for the case  $b = 0$  that is used in the present paper.

As a novelty, the Regular Convergence Lemma (Lemma 2.1) yields a direct argument for a large class of homogeneous *distributions*: recall that  $u \in \mathcal{D}'(\mathbb{R}^n)$  is homogeneous of degree  $a \in \mathbb{C}$  if

$$\langle u, \varphi \rangle = t^{n+a} \langle u, \varphi(t \cdot) \rangle, \quad \forall t > 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n). \quad (2.24)$$

When  $u \in \mathcal{S}'(\mathbb{R}^n)$  this extends to  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  by closure. This applies to the Littlewood–Paley decomposition  $1 = \sum_{j=0}^\infty \Phi_j$ , where  $\Phi_j(\xi) = \Phi(2^{-j}\xi)$  for  $j \geq 1$  and a fixed  $\Phi \in C^\infty$  (namely  $\Phi = \Psi(|\cdot|) - \Psi(2|\cdot|)$ ), so (2.24) gives directly

$$2^{ja} \Phi_j(D)u(x) = \langle u, 2^{j(a+n)} \check{\Phi}(2^j x - 2^j \cdot) \rangle = \Phi(D)u(2^j x). \quad (2.25)$$

Therefore  $2^{j(\frac{n}{p}+\operatorname{Re} a)}\|\Phi_j(D)u\|_p = \|\Phi(D)u\|_p$ , which is a constant independent of  $j$ . This can be exploited if  $u$  is assumed to have the origin as the only singularity:

**Proposition 2.8.** *Let  $u \in \mathcal{D}'(\mathbb{R}^n)$  be  $C^\infty$  on  $\mathbb{R}^n \setminus \{0\}$  and homogeneous of degree  $a \in \mathbb{C}$  there, ie (2.24) holds for all  $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ .*

*Then  $u$  is locally at  $x = 0$  in  $B_{p,\infty}^{\frac{n}{p}+\operatorname{Re} a}(\mathbb{R}^n)$  for  $0 < p \leq \infty$ . If  $-n < \operatorname{Re} a < 0$  it holds for  $-\frac{n}{\operatorname{Re} a} < p \leq \infty$  that  $u \in B_{p,\infty}^{\frac{n}{p}+\operatorname{Re} a}(\mathbb{R}^n)$ ; this holds also for  $p = \infty$  if  $\operatorname{Re} a = 0$ . The Besov space conclusions are sharp with respect to  $s$  and  $q$ , unless  $u$  is a homogenous polynomial (which is the only case in which  $u \in C^\infty(\mathbb{R}^n)$ ).*

*Proof.* The function  $D^\alpha u$  on  $\mathbb{R}^n \setminus \{0\}$  acts on  $\varphi$  like  $t^{a-|\alpha|} D^\alpha u(t^{-1}x)$ . Hence  $D^\alpha u$  has degree  $a - |\alpha|$ , and  $t = |x|$  entails  $|D^\alpha u(x)| \leq c_\alpha |x|^{\operatorname{Re} a - |\alpha|}$  for  $x \neq 0$ , all  $|\alpha| \geq 0$ .

If  $u \in C^\infty(\mathbb{R}^n)$ , the homogeneity of  $D^\alpha u$  gives  $D^\alpha u \equiv 0$  for  $\operatorname{Re} a - |\alpha| < 0$  (otherwise  $D^\alpha u$  would be discontinuous at  $x = 0$ ), and that  $D^\alpha u(0) = 0$  for  $\operatorname{Re} a - |\alpha| > 0$ . Therefore Taylor's formula gives at once that  $u \equiv 0$  if  $\operatorname{Re} a \notin \mathbb{N}_0$ , or else that  $u$  is a homogeneous polynomial (and  $a \in \mathbb{N}_0$ ).

The homogeneity and smoothness on  $\mathbb{R}^n \setminus \{0\}$  together imply that  $u \in \mathcal{S}'$  with  $\mathcal{F}u$  in  $C^\infty(\mathbb{R}^n \setminus \{0\})$ . This is known, cf [Hör85, Thm 7.1.18], but easy to see with a few ideas used here anyway: for  $\chi \in C_0^\infty(\mathbb{R}^n)$ ,  $\chi(0) = 1$ , one has  $u = \chi u + (1 - \chi)u$ , where the second term is in  $\mathcal{O}_M$  by the above, ie  $u \in \mathcal{E}' + \mathcal{O}_M \subset \mathcal{S}'$ . And  $\xi^\alpha D^\beta \hat{u} = \mathcal{F} D^\alpha((-x)^\beta u)$  is in  $\mathcal{F}(\mathcal{E}' + L_1) \subset C^0$  for  $|\alpha| > a + |\beta| + n$ , so  $\hat{u}$  is  $C^\infty$  for  $x \neq 0$ .

By the Paley–Wiener–Schwartz Theorem  $\Phi_j(D)(\chi u) \in \mathcal{S}(\mathbb{R}^n)$ . In particular for  $j = 0$  this gives  $\|\Phi_0(D)(\chi u)\|_p < \infty$ , while for  $j \geq 1$  it follows from (2.24) ff that, cf (2.25),

$$\Phi_j(D)(\chi u)(x) = \langle u, \chi 2^{jn} \check{\Phi}(2^j x - 2^j \cdot) \rangle = 2^{-ja} \Phi(D)(\chi(2^{-j} \cdot)u)(2^j x). \quad (2.26)$$



Here  $\chi(2^{-j}\cdot)$  is handled with Lemma 2.1, for since  $\Phi = 0$  near  $\text{sing supp } \hat{u} = \{0\}$ ,

$$(2\pi)^{-n} \Phi(\hat{u} * (2^{jn} \hat{\chi}(2^j \cdot))) \rightarrow \chi(0) \Phi \hat{u} \quad \text{in } C_0^\infty(\mathbb{R}^n). \quad (2.27)$$

Then the continuity of the embedding  $\mathcal{S} \hookrightarrow L_p$  and of the quasi-norm  $\|\cdot\|_p$  gives

$$\lim_{j \rightarrow \infty} 2^{j(\frac{n}{p} + \text{Re} a)} \|\Phi_j(D)(\chi u)\|_p = \|\Phi(D)u\|_p < \infty. \quad (2.28)$$

Hence  $\chi u \in B_{p,\infty}^{\frac{n}{p} + \text{Re} a}(\mathbb{R}^n)$  for all  $p$ . Note that the right hand side is zero if and only if  $\Phi \hat{u} \equiv 0$ , that by the homogeneity of  $\hat{u}$  is equivalent to  $\text{supp } \hat{u} \subset \{0\}$ , that holds if and only if  $u$  is a polynomial.

For  $-n < \text{Re} a < 0$  and some  $p \in ]-\frac{n}{\text{Re} a}, \infty] \subset ]1, \infty]$ , note first that

$$|x|^{\text{Re} a} \text{ is in } L_p \text{ for } |x| > 1 \iff p \text{Re} a < -n. \quad (2.29)$$

Since  $L_p * L_1 \subset L_p$  for  $p \geq 1$ , it follows that  $\check{\Phi}_0 * u$  belongs to  $L_p + \mathcal{S} \subset L_p$ . Likewise  $\check{\Phi}_0 * u \in L_\infty$  for  $\text{Re} a = 0$ , for  $u$  is bounded for  $|x| > 1$  by the first part of this proof. Now (2.25) gives that  $2^{j(\frac{n}{p} + \text{Re} a)} \|\Phi_j(D)u\|_p$  equals  $\|\Phi(D)u\|_p$ , which is finite since  $\Phi \hat{u} \in C_0^\infty$ . Therefore  $u \in B_{p,\infty}^{\frac{n}{p} + \text{Re} a}(\mathbb{R}^n)$ .

Because  $L_p \supset B_{p,\infty}^{\frac{n}{p} + \text{Re} a}$  for  $\text{Re} a + \frac{n}{p} > 0$ , the range for  $p$  is sharp, up to the end point  $p = -n/\text{Re} a$ , by (2.29). Since the other Besov space conclusions follow from identities, the spaces  $B_{p,\infty}^{\frac{n}{p} + \text{Re} a}$  are optimal (unless  $u$  is a polynomial).  $\square$

*Remark 2.9.* By Proposition 2.8,  $\frac{P(x)}{Q(x)} \in B_{p,\infty}^{\frac{n}{p}}(\mathbb{R}^n; \text{loc})$ ,  $0 < p \leq \infty$ , for two homogeneous polynomials  $P, Q$  both of degree  $a \geq 1$  such that  $Q(x) = 0$  only for  $x = 0$ . In case  $P \neq Q$  are real and  $n \geq 2$ , this has a special singularity since every neighbourhood of the origin is mapped onto the proper interval  $[\min_{|x|=1} \frac{P}{Q}, \max_{|x|=1} \frac{P}{Q}]$ . But the obtained Besov regularity  $B_{p,\infty}^{\frac{n}{p}}$  is the same as the well-known one for simple jump discontinuities across a hyperplane.

Invoking Lemma 2.7, the above analysis now leads to results for homogeneous distributions that are constant in  $n - k$  variables. A local version is given with optimal results for  $1 \leq p < \infty$ .

**Proposition 2.10.** *If  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$  is an open set with  $0 \in \Omega$  and the variables are split as  $x = (x', x'')$  for  $x' = (x_1, \dots, x_k)$ ,  $x'' = (x_{k+1}, \dots, x_n)$ , then it holds for every  $u(x')$  in  $\mathcal{D}'(\mathbb{R}^k)$  that is homogeneous of degree  $a \in \mathbb{C}$  and  $C^\infty$  outside of the origin that*

$$f(x) = r_\Omega[u(x') \otimes 1(x'')] \quad (2.30)$$

*belongs to  $B_{p,\infty}^s(\overline{\Omega})$  for  $s \leq \frac{k}{p} + \text{Re} a$ , except possibly for  $p = \infty$  if  $\text{Re} a = 0$ . For  $p \geq 1$  this result is sharp with respect to  $s$ .*

*Proof.* By Proposition 2.8, it follows from (2.16)–(2.17) that  $v(x) = (\varphi_1(x')u(x')) \otimes \varphi_2(x'')$  is in  $B_{p,\infty}^{\frac{k}{p} + \text{Re} a}(\mathbb{R}^n)$  for  $\frac{k}{p} + \text{Re} a \neq 0$  when the  $\varphi_j$  are both in  $C_0^\infty$ . For  $\text{Re} a = 0$  this excludes  $p = \infty$ , while for  $\text{Re} a < 0$  a gap is left at  $p_0 = k/(-\text{Re} a)$ , but this can be closed by Hölder's inequality, for if  $\frac{1}{p_0} = \frac{1}{2p_1} + \frac{1}{2p_2}$  for some exponents

$p_1 < p_0 < p_2$ , then each  $j \geq 0$  yields  $\|v_j\|_{p_0} \leq \prod_{m=1,2} (2^{j(\frac{k}{p_m} + \operatorname{Re} a)} \|v_j\|_{p_m})^{\frac{1}{2}} \leq \prod \|v\|_{B_{p_m,\infty}^{\frac{k}{p_m} + \operatorname{Re} a}}^{1/2}$ . Taking  $\varphi_1 \otimes \varphi_2$  equal to 1 on  $\Omega$  one finds, with the mentioned exception  $p = \infty$  for  $\operatorname{Re} a = 0$ ,

$$f = r_\Omega v \in B_{p,\infty}^s(\overline{\Omega}) \quad \text{for } s = \frac{k}{p} + \operatorname{Re} a, \quad 0 < p \leq \infty. \quad (2.31)$$

Conversely, if  $f$  is in this space for some  $s$ , it holds that  $w = (\theta_1 \otimes \theta_2)f \in B_{p,\infty}^s(\mathbb{R}^n)$  when the  $\theta_j \in C_0^\infty$  are supported sufficiently close to the origin. The support of  $\Phi_j'(\xi')\Phi_0''(\xi'')\hat{w}$  intersects that of  $\Phi_k(\xi)$  only for  $|j-k| \leq 2$ , so for  $p \geq 1$  the convolution result  $L_p * L_1 \subset L_p$  gives  $\|\Phi_j'(D')\Phi_0''(D'')w\|_p \leq c \sum_{|h| \leq 2} \|\Phi_{j+h}(D)w\|_p$ . Consequently

$$\begin{aligned} \sup_{j \geq 0} 2^{sj} \|\Phi_0''(D'')\theta_2\|_{L_p(\mathbb{R}^{n-2})} \|\Phi_j'(D')(\theta_1 f)\|_{L_p(\mathbb{R}^2)} \\ \leq c \sup_{j \geq 0} \sum_{|h| \leq 2} 2^{sj} \|\Phi_{j+h}(D)w\|_p \leq c_1 \|w\|_{B_{p,\infty}^s}. \end{aligned} \quad (2.32)$$

Taking  $\theta_2$  positive yields  $\hat{\theta}_2(0) = \int \theta_2 \neq 0$ , so  $\|\Phi_0''(D'')\theta_2\|_p > 0$  and as a result of this  $\|\theta_1 f\|_{B_{p,\infty}^s(\mathbb{R}^2)} < \infty$ . Then Proposition 2.8 gives  $s \leq \frac{k}{p} + \operatorname{Re} a$ .  $\square$

Since  $\delta_0$  has degree  $-n$  on  $\mathbb{R}^n$ , it is a special case that, for  $0 \in \Omega$ ,  $x = (x', x_n)$ ,

$$f(x) = 1(x') \otimes \delta_0(x_n) \quad \text{is in } B_{p,\infty}^{\frac{1}{p}-1}(\overline{\Omega}), \quad 0 < p \leq \infty. \quad (2.33)$$

### 3. THE GENERAL PARAMETRIX CONSTRUCTION

**3.1. An abstract framework.** For the applicability's sake Theorem 3.2 below is proved in a general set-up. If desired, the reader may think of the spaces  $X_p^s$  as  $H_p^s(\overline{\Omega})$  and consider  $A$  to be an elliptic operator like  $(\frac{-\Delta}{\mathfrak{y}_0})$  etc. The concepts in Section 1.3 are used freely, in particular this is so for parameter domains.

In the following five axioms,  $n \in \mathbb{N}$  and  $d \in \mathbb{R}$  are two fixed numbers, playing the role of the dimension and the order of the linear operator  $A$ , respectively, and  $I$  denotes the identity map:

- (I) Two scales  $X_p^s$  and  $Y_p^s$  of vector spaces are given with  $(s, p)$  in a common parameter set  $\mathbb{S} \subset \mathbb{R} \times ]0, \infty]$ . In the  $X_p^s$ -scale there are the usual simple, Sobolev and finite-measure embeddings; ie for  $(s, p), (t, r) \in \mathbb{S}$ ,

$$X_p^s \subset X_p^{s-\varepsilon} \quad \text{when } \varepsilon > 0, \quad (3.1)$$

$$X_p^s \subset X_r^t \quad \text{when } s \geq t \quad \text{and} \quad s - \frac{n}{p} = t - \frac{n}{r}, \quad (3.2)$$

$$X_p^s \subset X_r^s \quad \text{when } p \geq r. \quad (3.3)$$

- (II) There is a linear map  $A := A_{(s,p)}$ , with parameter domain  $\mathbb{D}(A) \subset \mathbb{S}$ ,

$$A: X_p^s \rightarrow Y_p^{s-d}, \quad (s, p) \in \mathbb{D}(A). \quad (3.4)$$

There is also for all  $(s, p) \in \mathbb{D}(A)$ , a linear map  $\tilde{A}: Y_p^{s-d} \rightarrow X_p^s$  such that

$$\mathcal{R} := I_{X_p^s} - \tilde{A}A \quad \text{has range in} \quad \bigcap_{(s,p) \in \mathbb{D}(A)} X_p^s. \quad (3.5)$$

Inclusions  $\bigcup_{\mathbb{D}(A)} X_p^s \subset \mathcal{X}$  and  $\bigcup_{\mathbb{D}(A)} Y_p^{s-d} \subset \mathcal{Y}$  hold for some vector spaces  $\mathcal{X}, \mathcal{Y}$ ; and for  $(s, p), (t, r) \in \mathbb{D}(A)$  there is a commutative diagram

$$\begin{array}{ccc} X_p^s \cap X_r^t & \xrightarrow{I} & X_p^s \\ I \downarrow & & \downarrow A_{(s,p)} \\ X_r^t & \xrightarrow{A_{(t,r)}} & \mathcal{Y}. \end{array} \quad (3.6)$$

Likewise  $\tilde{A}$  should be unambiguously defined on  $Y_p^{s-d} \cap Y_r^{t-d}$ .

- (III) There is a non-linear operator  $\mathcal{N}$ , with parameter domain  $\mathbb{D}(\mathcal{N}) \subset \mathbb{S}$ , which for every  $(s_0, p_0)$  in  $\mathbb{D}(\mathcal{N})$  and every  $u \in X_{p_0}^{s_0}$  has a linearisation  $B_u$ , ie  $\mathcal{N}(u) = -B_u(u)$ , where  $B_u$  is a linear map

$$B_u: X_p^s \rightarrow Y_p^{s-d+\delta(s,p)} \quad \text{with} \quad \mathbb{D}(B_u) \supset \mathbb{D}(\mathcal{N}). \quad (3.7)$$

For  $(s, p), (t, r) \in \mathbb{D}(B_u)$  there is a commutative diagram analogous to (3.6) for  $B_u$  (hence for  $\mathcal{N}$ ).

- (IV) For  $u$  as in (III), the domain  $\mathbb{D}(A) \cap \mathbb{D}(B_u)$  is connected with respect to the metric  $\text{dist}((s, p), (t, r))$  given by  $((s-t)^2 + (\frac{p}{r} - \frac{r}{p})^2)^{1/2}$ .
- (V) For  $u$  as in (III), the function  $\delta(s, p)$  satisfies

$$(s + \delta(s, p), p) \in \mathbb{D}(A) \quad \text{for every} \quad (s, p) \in \mathbb{D}(A) \cap \mathbb{D}(B_u), \quad (3.8)$$

$$\inf\{\delta(s, p) \mid (s, p) \in K\} > 0 \quad \text{for every} \quad K \Subset \mathbb{D}(A) \cap \mathbb{D}(B_u). \quad (3.9)$$

For the proof of Theorem 3.2 below it is unnecessary to assume that the embeddings in (I) should hold for the  $Y_p^s$  spaces too (although they often do so in practice). As it stands (I) is easier to verify in applications to parabolic boundary problems; cf Remark 8.3 below.

For  $X_p^s = H_p^s(\overline{\Omega})$  it is natural to let  $\mathbb{S} = \mathbb{R} \times ]1, \infty[$ ; the  $L_2$ -theory comes out for  $\mathbb{S} = \mathbb{R} \times \{2\}$ . Besov spaces  $B_{p,q}^s$  would often require  $q$  to be fixed and  $\mathbb{S} = \mathbb{R} \times ]0, \infty]$ . Anyhow  $\mathcal{X} = \mathcal{D}'(\Omega)$  could be a typical choice. Continuity of  $A$  and  $\tilde{A}$  is not required (although both will be bounded in most applications).

Suppressing  $(s, p)$  in  $A$  is harmless in the sense that  $A$  by (3.6) is a well-defined map with domain  $\bigcup_{\mathbb{D}(A)} X_p^s$  in  $\mathcal{X}$ ; it is linear only on each ‘fibre’  $X_p^s$ . Similarly  $\tilde{A}$  is a map on  $\bigcup_{\mathbb{D}(A)} Y_p^{s-d}$ . Moreover,  $A$  eg extends to a linear map on the algebraic direct sum  $\bigoplus X_p^s \subset \mathcal{X}$  if and only if (when  $'$  indicates finitely many non-trivial vectors)

$$0 = \sum'_{\mathbb{D}(A)} v_{(s,p)} \implies \sum_{\mathbb{D}(A)} A_{(s,p)}(v_{(s,p)}) = 0.$$

By (3.6) ff,  $\mathcal{R}$  may be thought of as an operator from  $\bigcup_{\mathbb{D}(A)} X_p^s$  to  $\bigcap_{\mathbb{D}(A)} X_p^s$ .

For brevity the arguments  $s_0, p_0$  are suppressed in the function  $\delta$ . By (III), the map  $\mathcal{N}$  sends  $X_p^s$  into  $Y_p^{s-d+\delta(s,p)}$  for each  $(s, p)$  in  $\mathbb{D}(\mathcal{N})$  (since  $\mathbb{D}(\mathcal{N}) \subset \mathbb{D}(B_u)$ )

for every  $u$  in  $X_p^s$ . This fact will be used tacitly. Note that  $\delta(s, p) > 0$  by (3.9), so (III) implies that  $\mathcal{N}(u)$  has  $B_u$  as a moderate linearisation with  $\omega = d - \delta(s, p)$ , according to Definition 1.2.

Via the transformation  $(s, p) \mapsto (\frac{n}{p}, s)$ , the reader should constantly think of  $\mathbb{D}(A)$ ,  $\mathbb{D}(\mathcal{N})$  and  $\mathbb{D}(B_u)$  as subsets of  $[0, \infty[ \times \mathbb{R}$ . In the examples the boundary of  $\mathbb{D}(\mathcal{N})$  (or a part thereof) often consists of the  $(s_0, p_0)$  for which  $\delta \equiv 0$ , so it may seem natural to require  $\mathbb{D}(\mathcal{N})$  to be open in  $[0, \infty[ \times \mathbb{R}$ . However, such an assumption is avoided because it is unnecessary and potentially might exclude application to weak solutions of certain problems; cf the below Section 6.

The function  $\delta$  is in practice often constant with respect to  $(s, p)$ , but depending effectively on  $(s_0, p_0)$ ; cf Remark 1.3. When this is the case and furthermore  $\mathcal{N}$  has a natural parameter domain  $\mathbb{D}(\mathcal{N})$  on which  $\delta$  can take both positive and negative values, it is natural to use

$$\mathbb{D}(\mathcal{N}, \delta) = \{(s_0, p_0) \in \mathbb{D}(\mathcal{N}) \mid \delta > 0\} \quad (3.10)$$

as the parameter domain of  $\mathcal{N}$ , instead of  $\mathbb{D}(\mathcal{N})$ . Then  $\mathcal{N}$  will be  $A$ -moderate on the domain

$$\mathbb{D}(A, \mathcal{N}) = \mathbb{D}(A) \cap \mathbb{D}(\mathcal{N}, \delta). \quad (3.11)$$

With  $\sigma(s, p) := d - \delta(s, p)$ , it is clear that  $\mathbb{D}(A, \mathcal{N})$  is a generalisation of the domain  $\mathbb{D}(\mathcal{A}, Q)$  introduced for the model problem in (1.29). We now return to this.

**Example 3.1.** To elucidate (I)–(V) above, one may in (1.3) set  $A = \left(\frac{-\Delta}{\gamma_0}\right)$  and  $X_p^s = B_{p,q}^s(\overline{\Omega})$ , whereby  $q \in ]0, \infty]$  is kept fixed. For the operator  $\tilde{A}$  there is a parametrix of  $A$  belonging to the Boutet de Monvel calculus (cf Section 7.1 below). Using  $L_u$  from (1.15),  $B_u$  and  $Y_p^s$  are taken as

$$B_u v = \begin{pmatrix} L_u v \\ 0 \end{pmatrix}, \quad Y_p^{s-2} = B_{p,q}^{s-2}(\overline{\Omega}) \oplus B_{p,q}^{s-\frac{1}{p}}(\Gamma). \quad (3.12)$$

For any  $\varepsilon \in ]0, 1[$  it is possible to take  $\delta(s, p)$  as the constant function

$$\delta(s, p) = \begin{cases} 1 & \text{for } s_0 > \frac{n}{p_0}, \\ 1 - \varepsilon & \text{for } s_0 = \frac{n}{p_0}, \\ s_0 - \frac{n}{p_0} + 1 & \text{for } \frac{n}{p_0} > s_0 > \frac{n}{p_0} - 1. \end{cases} \quad (3.13)$$

See the below Theorem 5.7. As mentioned in Remark 5.10, this theorem and Corollary 5.9 also gives the parameter domains, for any fixed  $u \in X_{p_0}^{s_0}$ ,

$$\mathbb{D}(A) = \{(s, p) \mid s > \frac{1}{p} + (n-1)(\frac{1}{p} - 1)_+\} = \mathbb{D}_1, \quad (3.14)$$

$$\mathbb{D}(\mathcal{N}) = \{(s, p) \mid s > \frac{1}{2} + (\frac{n}{p} - \frac{n}{2})_+\}, \quad (3.15)$$

$$\mathbb{D}(\mathcal{N}, \delta) = \{(s, p) \mid s > \frac{1}{2} + (\frac{n}{p} - \frac{3}{2} + \frac{1}{2} \llbracket n=2 \rrbracket)_+\} = \mathbb{D}(A, \mathcal{N}), \quad (3.16)$$

$$\mathbb{D}(B_u) = \{(s, p) \mid s > 1 - s_0 + (\frac{n}{p} + \frac{n}{p_0} - n)_+\}. \quad (3.17)$$

Being isometric to a polygon in  $[0, \infty[ \times \mathbb{R}$ , the set  $\mathbb{D}(A) \cap \mathbb{D}(B_u)$  clearly satisfies (IV); when  $(s_0, p_0) \in \mathbb{D}(A) \cap \mathbb{D}(B_u)$ , then condition (V) may be verified directly from (3.13).

**3.2. The Parametrix Theorem.** Using the above abstract framework, it is now possible to establish a main result of the article in a widely applicable version.

**Theorem 3.2.** *Let  $X_p^s$ ,  $Y_p^s$  and the mappings  $A$  and  $\mathcal{N}$  be given such that conditions (I)–(V) above are satisfied.*

(1) *For every*

$$u \in X_{p_0}^{s_0} \quad \text{with } (s_0, p_0) \in \mathbb{D}(A) \cap \mathbb{D}(\mathcal{N}) \quad (3.18)$$

*the parametrix  $P^{(N)} = \sum_{k=0}^{N-1} (\tilde{A}B_u)^k$  is for every  $N \in \mathbb{N}$  a linear operator*

$$P^{(N)} : X_p^s \rightarrow X_p^s \quad \text{for all } (s, p) \in \mathbb{D}(A) \cap \mathbb{D}(B_u) =: \mathbb{D}_u. \quad (3.19)$$

*And for every  $(s', p')$ ,  $(s'', p'') \in \mathbb{D}_u$  there exists  $N' \in \mathbb{N}$  such that the “error term”  $(\tilde{A}B_u)^N$  is a linear map*

$$(\tilde{A}B_u)^N : X_{p'}^{s'} \rightarrow X_{p''}^{s''} \quad \text{for } N \geq N'. \quad (3.20)$$

(2) *If some  $u$  fulfils (3.18) and solves the equation*

$$Au + \mathcal{N}(u) = f \quad (3.21)$$

$$\text{with data } f \in Y_r^{t-d} \quad \text{for some } (t, r) \in \mathbb{D}_u, \quad (3.22)$$

*one has for every  $N \in \mathbb{N}$  the parametrix formula*

$$u = P^{(N)}(\tilde{A}f + \mathcal{R}u) + (\tilde{A}B_u)^N u. \quad (3.23)$$

*And consequently  $u \in X_r^t$  too.*

*Proof.* For arbitrary  $(s, p) \in \mathbb{D}_u$ , one can use (II) and (3.8) to see that  $\tilde{A}$  is defined on  $Y_p^{s-d+\delta(s,p)}$ , hence that  $\tilde{A}B_u$  is a well defined composite

$$X_p^s \xrightarrow{B_u} Y_p^{s-d+\delta(s,p)} \xrightarrow{\tilde{A}} X_p^{s+\delta(s,p)}. \quad (3.24)$$

Since  $X_p^{s+\delta} \hookrightarrow X_p^s$  by (I), the operator  $\tilde{A}B_u$  is of order 0 on  $X_p^s$ ; hence  $P^{(N)} := \sum_{j=0}^{N-1} (\tilde{A}B_u)^j$  is a linear map  $X_p^s \rightarrow X_p^s$ . This shows the claim on  $P^{(N)}$ .

Concerning  $(\tilde{A}B_u)^N$ , there is, by (IV), a continuous map  $k : I \rightarrow \mathbb{D}_u$ , with  $I = [a, b]$ , such that

$$k(a) = (s', p'), \quad k(b) = (s'', p''). \quad (3.25)$$

Clearly  $\delta_k := \inf \{ \delta(s, p) \mid (s, p) \in k(I) \} > 0$  by (V), and for  $(s, p) \in k(I)$

$$X_p^s \xrightarrow{\tilde{A}B_u} X_p^{s+\delta(s,p)} \hookrightarrow X_p^{s+\delta_k}. \quad (3.26)$$

With  $X_{k(\tau)} := X_p^s$  when  $k(\tau) = (s, p)$ , let  $M := \sup T$  for

$$T = \{ \tau \in I \mid \exists N \in \mathbb{N} : (\tilde{A}B_u)^N(X_{p'}^{s'}) \subset X_{k(\tau)} \}. \quad (3.27)$$

Then  $a \leq M \leq b$  since  $\tilde{A}B_u(X_{p'}^{s'}) \subset X_{p'}^{s'+\delta} \subset X_{k(a)}$ . It now suffices to show that  $b \in T$ , for  $(\tilde{A}B_u)^N(X_{p'}^{s'}) \subset X_{k(b)} = X_{p''}^{s''}$  for some  $N \in \mathbb{N}$  then; and  $(\tilde{A}B_u)^N$  equals  $(\tilde{A}B_u)^{N-N'}(\tilde{A}B_u)^{N'}$  for  $N > N'$ , so the full claim on  $\tilde{A}B_u$  would follow because (3.26) shows that  $(\tilde{A}B_u)^{N-N'}$  is of order 0 on  $X_{p''}^{s''}$ .

For one thing  $M \in T$ : by continuity of  $k$  there is a  $\tau' < M$  in  $T$  such that, when  $|\cdot|$  denotes the Euclidean norm on  $\mathbb{R}^2$  and the isometry  $(s, p) \leftrightarrow (\frac{n}{p}, s)$  is suppressed,

$$|k(\tau') - k(M)| < \delta_k/2 \quad (3.28)$$

and  $(\tilde{A}B_u)^{N-1}(X_{p'}^{s'}) \subset X_{k(\tau')}$  for some  $N$ . But by (3.26) this entails that  $(\tilde{A}B_u)^N(X_{p'}^{s'})$  is a subset of a space with upper index at least  $\delta_k$  higher than that of  $X_{k(\tau')}$ , so the embeddings in (I) show that  $(\tilde{A}B_u)^N(X_{p'}^{s'})$  is contained in any space in the intersection of  $\mathbb{S}$  and a convex polygon; cf the dashed line in Figure 1 below. It follows that  $(\tilde{A}B_u)^N(X_{p'}^{s'})$  is contained in every  $X_p^s$  lying in  $\mathbb{S}$  and fulfilling

$$|k(\tau') - (s, p)| < \delta_k/\sqrt{2}, \quad (3.29)$$

so in particular  $(\tilde{A}B_u)^N(X_{p'}^{s'}) \subset X_{k(M)}$  is found from (3.28); whence  $M \in T$ .

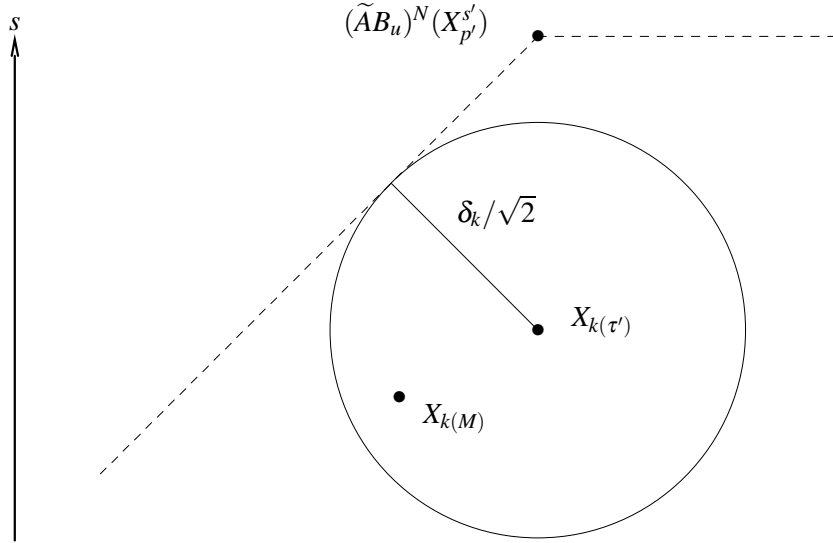


FIGURE 1. The  $(\frac{n}{p}, s)$ -plane with the ball in (3.29) and a polygon of spaces containing  $(\tilde{A}B_u)^N(X_{p'}^{s'})$ .

Secondly,  $M = b$  follows from  $k(I)$ 's connectedness: assuming that  $\tau \in ]M, b]$  exists, the curve  $k(\tau)$  would for some  $\tau > M$  lie in the open  $\frac{\delta_k}{2}$ -ball around  $k(M)$ . Then  $|k(\tau) - k(M)| \leq \frac{\delta_k}{2} < \frac{\delta_k}{\sqrt{2}}$  would hold, so that the proved fact  $M \in T$  would imply (as above) that  $(\tilde{A}B_u)^N(X_{p'}^{s'}) \subset X_{k(\tau)}$ , contradicting that  $\tau \notin T$ .

According to (II), (III) and the assumptions in the theorem, the mapping  $\tilde{A}$  has the same meaning on both sides of (3.21), regardless of whether one refers to  $Y_{p_0}^{s_0-d}$  or to  $Y_r^{t-d}$  (on the left and the right hand sides, respectively). Therefore (3.5) and the assumption  $(s_0, p_0) \in \mathbb{D}(\mathcal{N})$  entail

$$(I - \mathcal{R})u - \tilde{A}B_u u = \tilde{A}f. \quad (3.30)$$

For the given  $u$  and  $f$ , it follows by calculation of the telescopic sum that

$$P^{(N)}(I - \tilde{A}B_u)u = \sum_{j=0}^{N-1} (\tilde{A}B_u)^j (I - \tilde{A}B_u)u = (I - (\tilde{A}B_u)^N)u, \quad (3.31)$$

and  $P^{(N)}$  has the same meaning when applied to both sides of (3.30). Therefore (3.30), (3.31) yield (3.23).

Note that the term  $P^{(N)}(\tilde{A}f + \mathcal{R}u)$  in (3.23) is in  $X_r^t$  in view of (3.5) and the proved fact that  $P^{(N)}$  has order 0 on  $X_r^t$ . By (3.20) also  $(\tilde{A}B_u)^N u$  is in  $X_r^t$ , so this holds for every given solution  $u$  too.  $\square$

Applications of Theorem 3.2 to systems of elliptic boundary problems are developed in Section 7 below for non-linear terms of product type. In this context (3.8) in (V) is redundant, for with  $\mathbb{D}(A)$  equal to one of the standard domains  $\mathbb{D}_k$  it is for any  $\eta > 0$  clear that  $(s + \eta, p)$  belongs to  $\mathbb{D}(A)$  when  $(s, p)$  does so. But (3.8) is inserted in preparation for applications to other non-linearities, like  $|u|^a$  with non-integer  $a > 0$ ; and to parabolic problems, cf Remark 8.3 below.

**3.3. A solvability result.** As an addendum to the Parametrix Theorem, it is used for the solvability in eg problem (IR) and Theorem 8.1 below that bilinear perturbations of linear homeomorphisms always give well-posed problems locally, ie for sufficiently small data.

It should be folklore how to obtain this from the fixed-point theorem of contractions. The proof extends to any quasi-Banach space  $X$  for which  $\|\cdot\|^\lambda$  is subadditive for some  $\lambda \in ]0, 1]$ , for  $d(x, y) = \|x - y\|^\lambda$  is a complete metric on  $X$  then. (For  $B_{p,q}^s$  and  $F_{p,q}^s$  the existence of  $\lambda$  is easy to see, cf (2.6) ff.) In lack of a reference the next result is given, including details for the lesser known quasi-Banach space case.

**Proposition 3.3.** *Let  $A: X \rightarrow Y$  be a linear homeomorphism between two quasi-Banach spaces and  $B: X \oplus X \rightarrow Y$  be a bilinear bounded map. When  $\|\cdot\|^\lambda$  is subadditive for some  $\lambda \in ]0, 1]$  and  $y \in Y$  fulfills*

$$\|A^{-1}y\|^\lambda < \|A^{-1}B\|^{-1} 4^{-1/\lambda}, \quad (3.32)$$

*then the ball  $\|x\|^\lambda < \|A^{-1}B\|^{-1} 2^{-1/\lambda}$  contains a unique solution of the equation*

$$Ax + B(x) = y. \quad (3.33)$$

*This solution depends continuously on  $y$  in the ball (3.32).*

*Proof.* When  $R := A^{-1}$ , the equation is equivalent to  $x = Ry - RB(x) =: F(x)$ , where also  $RB =: B'$  is bilinear and  $\|B'\| \leq \|R\|\|B\|$ . Bilinearity gives

$$\|F(x) - F(z)\|^\lambda \leq \|B'(x, x - z)\|^\lambda + \|B'(x - z, z)\|^\lambda \leq \|B'\|^\lambda (\|x\|^\lambda + \|z\|^\lambda) \|x - z\|^\lambda, \quad (3.34)$$

so  $F$  is a contraction on the closed ball  $K_a = \{x \in X \mid \|x\| \leq a\}$  if  $a$  fulfills  $2\|B'\|^\lambda a^\lambda < 1$ . By the assumptions  $D := 1 - 4\|Ry\|^\lambda \|B'\|^\lambda > 0$ , so

$$\|Ry\|^\lambda + \|B'\|^\lambda t^2 < t \iff t \in \left] \frac{1 - \sqrt{D}}{2\|B'\|^\lambda}, \frac{1 + \sqrt{D}}{2\|B'\|^\lambda} \right[; \quad (3.35)$$

here the interval contains  $t = a^\lambda$ , when  $a^\lambda$  is sufficiently close to  $(2\|B'\|^\lambda)^{-1}$ . So, since  $x \in K_a$  implies  $\|F(x)\|^\lambda \leq \|Ry\|^\lambda + \|B'\|^\lambda a^{2\lambda}$ , it follows from (3.35) that  $F(x) \in K_a$ , ie  $F$  is a map  $K_a \rightarrow K_a$  for such  $a$ . Hence  $x = F(x)$  is uniquely solved in  $K_a$ . If also  $Ax' + B(x') = y'$  for some  $x' \in K_a$ ,

$$\|x - x'\|^\lambda \leq \|R(y - y')\|^\lambda + 2a^\lambda \|B'\|^\lambda \|x - x'\|^\lambda, \quad (3.36)$$

so  $d(x, x') \leq cd(Ry, Ry')$  for  $c = (1 - 2a^\lambda \|B'\|^\lambda)^{-1} < \infty$ . This gives the well-posedness in  $K_a$ , but with the leeway in the choice of  $a$  the proposition follows.  $\square$

#### 4. PRELIMINARIES ON PRODUCTS

A brief review of results on pointwise multiplication is given before the non-linear operators of product type are introduced in Sections 5 and 7 below.

**4.1. Generalised multiplication.** In practice non-linearities often involve multiplication of a non-smooth function and a distribution in  $\mathcal{D}' \setminus L_1^{\text{loc}}$ , as in  $u\partial_1 u$  when  $u \in H^{\frac{1}{2}+\varepsilon}$  for small  $\varepsilon > 0$ . Although it suffices for a mere construction of solutions to extend  $(u, v) \mapsto u \cdot v$  by continuity to a bounded bilinear form defined on  $H^s \times H^{-s}$  for some  $s > 0$ , the proof of the regularity properties will in general involve extensions to  $F_{p,q}^s \times F_{p,q}^{-s}$  for *several* exponents  $p$  and  $q$ . This would clearly cause a problem of consistency among the various extensions, and for  $q = \infty$  there would, moreover, not be density of smooth functions to play on. Commutative diagrams like (3.6) would then be demanding to verify for the product type operators, so a more unified approach to multiplication is preferred here.

Since a paper of L. Schwartz [Sch54] it has been known that products with a few reasonable properties cannot be everywhere defined on  $\mathcal{D}' \times \mathcal{D}'$ , and as a consequence many notions of multiplication exist, cf the survey [Obe92]. But for the present theory it is important to use a product  $\pi(\cdot, \cdot)$  that works well together with paramultiplication on  $\mathbb{R}^n$  and also allows a localised version  $\pi_\Omega$  to be defined on an open set  $\Omega \subset \mathbb{R}^n$ . A product  $\pi$  with these properties was analysed in [Joh95], cf also [RS96], and for the reader's sake a brief review is given.

The product  $\pi$  is defined on  $\mathbb{R}^n$  by simultaneous regularisation of both factors: for  $\psi_k(\xi) = \psi(2^{-k}\xi)$  with  $\psi \in C_0^\infty(\mathbb{R}^n)$  equal to 1 in a neighbourhood of  $\xi = 0$ ,

$$\pi(u, v) := \lim_{k \rightarrow \infty} (\psi_k(D)u) \cdot (\psi_k(D)v). \quad (4.1)$$

Here  $u$  and  $v \in \mathcal{S}'(\mathbb{R}^n)$ , and they are required to have the properties that this limit should both exist in  $\mathcal{D}'(\mathbb{R}^n)$  for *all*  $\psi$  of the specified type and be independent of the choice of  $\psi$ . ( $\psi_k(D)u := \mathcal{F}^{-1}(\psi_k \hat{u})$  etc.)

This formal definition is from [Joh95], but analogous limits have been known for a long time in the  $\mathcal{D}'$ -context. As shown in [Joh95, Sect. 3.1],  $\pi(u, v)$  coincides with the usual pointwise multiplication:

$$L_p^{\text{loc}}(\mathbb{R}^n) \times L_q^{\text{loc}}(\mathbb{R}^n) \xrightarrow{\cdot} L_r^{\text{loc}}(\mathbb{R}^n), \quad 0 \leq \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1, \quad (4.2)$$

$$\mathcal{O}_M(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \xrightarrow{\cdot} \mathcal{S}'(\mathbb{R}^n). \quad (4.3)$$



For later reference, the main tool for (4.2) and localisation to open sets  $\Omega$  is recalled from [Joh95, Prop. 3.7]: if either  $u$  or  $v$  vanishes in  $\Omega$ , then any  $\psi$  as in (4.1) gives

$$0 = \lim_{k \rightarrow \infty} r_\Omega(\psi_k(D)u \cdot \psi_k(D)v) \quad \text{in } \mathcal{D}'(\Omega). \quad (4.4)$$

When  $\pi(u, v)$  is defined, (4.4) implies  $\text{supp } \pi(u, v) \subset \text{supp } u \cap \text{supp } v$  (this is obvious for (4.2)–(4.3)). But as a consequence of Lemma 2.1, the limit in (4.4) exists in any case when one of the factors vanish in  $\Omega$ .

Using (4.4),  $\pi_\Omega(u, v)$  is *defined* for an arbitrary open set  $\Omega \subset \mathbb{R}^n$  on those  $u, v$  in  $\mathcal{D}'(\Omega)$  for which  $U, V \in \mathcal{S}'(\mathbb{R}^n)$  exist such that  $r_\Omega U = u$ ,  $r_\Omega V = v$  and

$$\pi_\Omega(u, v) := \lim_{k \rightarrow \infty} r_\Omega[(\psi_k(D)U) \cdot (\psi_k(D)V)] \quad \text{exists in } \mathcal{D}'(\Omega) \quad (4.5)$$

independently of  $\psi \in C_0^\infty(\mathbb{R}^n)$  with  $\psi \equiv 1$  near  $\xi = 0$ . Hereby  $\pi_\Omega$  is well defined, for (4.4) implies that the limit is independent of the ‘extension’  $(U, V)$ , hence the  $\psi$ -independence is so (cf [Joh95, Def 7.1]). But as  $\pi(U, V)$  need not be defined, it is clearly essential that  $r_\Omega$  is applied before passing to the limit.

**4.2. Boundedness of generalised multiplication.** Using (4.5), it is easy to see, and well known, that  $\pi_\Omega$  inherits boundedness from  $\pi$  on  $\mathbb{R}^n$  as follows:

**Proposition 4.1.** *Let each of the spaces  $E_0, E_1$  and  $E_2$  be either a Besov space  $B_{p,q}^s(\mathbb{R}^n)$  or a Lizorkin–Triebel space  $F_{p,q}^s(\mathbb{R}^n)$ , chosen so that  $\pi(\cdot, \cdot)$  is a bounded bilinear operator*

$$\pi: E_0 \oplus E_1 \rightarrow E_2. \quad (4.6)$$

*For the corresponding spaces  $E_k(\overline{\Omega}) := r_\Omega E_k$  over an arbitrary open set  $\Omega \subset \mathbb{R}^n$ , endowed with the infimum norm,  $\pi_\Omega$  is bounded*

$$\pi_\Omega(\cdot, \cdot): E_0(\overline{\Omega}) \oplus E_1(\overline{\Omega}) \rightarrow E_2(\overline{\Omega}). \quad (4.7)$$

In the result above it is a central question under which conditions (4.6) actually holds. This was almost completely analysed in [Joh95, Sect. 5] by means of paramultiplication. To prepare for the definition and analysis (further below) of the exact parilinearisation, this will now be recalled.

First, by using (2.4) and setting  $\Phi_j \equiv 0 \equiv \Psi_j$  for  $j < 0$ , the paramultiplication operators  $\pi_m(\cdot, \cdot)$  with  $m = 1, 2, 3$  (in the sense of M. Yamazaki [Yam86a, Yam86b, Yam88]), are defined for those  $f, g \in \mathcal{S}'(\mathbb{R}^n)$  for which the series below converge in  $\mathcal{D}'(\mathbb{R}^n)$ :

$$\pi_1(f, g) = \sum_{j=0}^{\infty} \Psi_{j-2}(D) f \Phi_j(D) g \quad (4.8a)$$

$$\pi_2(f, g) = \sum_{j=0}^{\infty} (\Phi_{j-1}(D) f \Phi_j(D) g + \Phi_j(D) f \Phi_j(D) g + \Phi_j(D) f \Phi_{j-1}(D) g) \quad (4.8b)$$

$$\pi_3(f, g) = \sum_{j=0}^{\infty} \Phi_j(D) f \Psi_{j-2}(D) g \quad (4.8c)$$

This applies to (4.1) by taking  $\psi_k = \Psi_k$ , for  $\Psi_k = \Phi_0 + \dots + \Phi_k$ , so that bilinearity gives that the limit on the right hand side of (4.1) equals  $\sum_{m=1,2,3} \pi_m(u, v)$ , whenever each  $\pi_m(u, v)$  exists — but this existence is easily analysed for each  $m$  by standard estimates. In fact  $\pi_1(f, g)$  and  $\pi_3(f, g)$  both exist for all  $f, g \in \mathcal{S}'(\mathbb{R}^n)$ , as observed in [MC97, Ch. 16], so  $\pi(u, v)$  is defined if and only if the second series  $\pi_2(u, v)$  is so. Finally the  $\psi$ -independence is established post festum; cf [Joh95, Sect. 6.4].

For convenience  $E_{p,q}^s$  will now denote a space which (for every value of  $(s, p, q)$ ) may be either a Besov or a Lizorkin–Triebel space on  $\mathbb{R}^n$ . It was proved in [Joh95, Thm. 4.2], albeit with (4.10b) and (4.11b) essentially covered by [Fra86b], that if

$$\|fg\|_{E_{p_2,q_2}^{s_2}} \leq c \|f\|_{E_{p_0,q_0}^{s_0}} \|g\|_{E_{p_1,q_1}^{s_1}} \quad (4.9)$$

holds for all Schwartz functions  $f$  and  $g$ , then

$$s_0 + s_1 \geq n\left(\frac{1}{p_0} + \frac{1}{p_1} - 1\right), \quad (4.10a)$$

$$s_0 + s_1 \geq 0. \quad (4.10b)$$

As a supplement to this, the following were also established there:

$$s_0 + s_1 = \frac{n}{p_0} + \frac{n}{p_1} - n \quad \text{implies} \quad \begin{cases} \frac{1}{q_0} + \frac{1}{q_1} \geq 1 \text{ in } BB^\bullet\text{-cases,} \\ \frac{1}{q_0} + \frac{1}{q_1} \geq 1 \text{ in } BF^\bullet\text{-cases;} \end{cases} \quad (4.11a)$$

$$s_0 + s_1 = 0 \quad \text{implies} \quad \frac{1}{q_0} + \frac{1}{q_1} \geq 1. \quad (4.11b)$$

The main interest lies in the  $BB^\bullet$ - and  $FF^\bullet$ -cases and the case with  $\max(s_0, s_1) > 0$  (for  $s_0 = s_1 = 0$  Hölder's inequality applies). In this situation the sufficiency of the above conditions was entirely confirmed by means of (4.8), cf the following version of [Joh95, Cor 6.12] for isotropic spaces:

**Theorem 4.2.** *When  $\max(s_0, s_1) > 0$ , then it holds in the  $BB^\bullet$ - and  $FF^\bullet$ -cases that  $E_{p_0,q_0}^{s_0}$  and  $E_{p_1,q_1}^{s_1}$  on  $\mathbb{R}^n$  are ‘multiplicable’ if and only if both (4.10a)–(4.10b) and (4.11a)–(4.11b) hold.*

The spaces that receive  $\pi(E_{p_0,q_0}^{s_0}, E_{p_1,q_1}^{s_1})$  were almost characterised in [Joh95], departing from at least 8 other necessary conditions, but the below Theorem 5.7 will imply what is needed in this direction.

*Remark 4.3.* To prepare for Theorem 5.7 below, a few estimates of the  $\pi_j$  are recalled. When  $\frac{1}{p_2} = \frac{1}{p_0} + \frac{1}{p_1}$ ,  $\frac{1}{q_2} = \frac{1}{q_0} + \frac{1}{q_1}$ , there is boundedness

$$\pi_1 : L_\infty \oplus B_{p,q}^s \rightarrow B_{p,q}^s \quad (4.12)$$

$$\pi_1 : B_{p_0,q_0}^{s_0} \oplus B_{p_1,q_1}^{s_1} \rightarrow B_{p_2,q_2}^{s_0+s_1} \quad \text{for } s_0 < 0, \quad (4.13)$$

$$\pi_2 : B_{p_0,q_0}^{s_0} \oplus B_{p_1,q_1}^{s_1} \rightarrow B_{p_2,q_2}^{s_0+s_1} \quad \text{for } s_0 + s_1 > \left(\frac{n}{p_2} - n\right)_+. \quad (4.14)$$

Since  $\pi_3(f, g) = \pi_1(g, f)$ , also  $\pi_3$  is covered by this. Analogous results hold for the Lizorkin–Triebel spaces, except that Lemma 2.4 for  $s_0 + s_1 > \left(\frac{n}{p_2} - n\right)_+$  entails

$$\pi_2 : F_{p_0,q_0}^{s_0} \oplus F_{p_1,q_1}^{s_1} \rightarrow F_{p_2,t}^{s_0+s_1} \quad \text{when} \quad \begin{cases} t \geq q_2 & \text{for } q_2 \geq p_2 \\ t > \frac{n}{n+s_0+s_1} & \text{for } q_2 < p_2. \end{cases} \quad (4.15)$$

These estimates all follow from the dyadic corona and ball criteria in a way that is standard by now, so details are omitted (the arguments can be found in a refined version for a special case in Proposition 4.5 below, cf also the proof of Lemma 2.7). They go back to the paradifferential estimates of M. Yamazaki [Yam86a], but in the simpler context of paramultiplication an account of the estimates may be found in eg [Joh95, Thm 5.1], though with (4.15) as a small improvement.

*Remark 4.4.* It is used in Section 8 below that multiplication cannot define a continuous map  $W_1^m \oplus W_1^m \rightarrow \mathcal{D}'$  when  $2m < n$ . When the range is a Besov space this follows on  $\mathbb{R}^n$  from (4.10a), but for the general statement an explicit proof should be in order. If  $\rho \in C_0^\infty$  is real and  $\rho_k(x) = \frac{1}{k} 2^{k(n-m)} \rho(2^k x)$ , it is easy to see that  $\|\rho_k|W_1^m\| = \mathcal{O}(\frac{1}{k}) \searrow 0$ . But for  $\varphi \in C_0^\infty$  non-negative with  $\varphi(0) = 1$ ,  $2m < n$  implies

$$\langle \rho_k^2, \varphi \rangle = k^{-2} 2^{k(n-2m)} \int \rho^2(y) \varphi(2^{-k}y) dy \rightarrow \infty. \quad (4.16)$$

This argument works for open sets  $\Omega \ni 0$  and extends to all  $\Omega \subset \mathbb{R}^n$  by translation.

**4.3. Extension by zero.** Having presented the product  $\pi(\cdot, \cdot)$  formally, the opportunity is taken to make a digression needed later.

In Section 6–7 the operators  $A$  and  $\tilde{A}$  of Section 3 will be realised through the Boutet de Monvel calculus of linear boundary problems, so it will be all-important to have commuting diagrams like (3.6) for the operators in the calculus. Avoiding too many details, the main step is to show that truncated pseudo-differential operators are defined independently of the spaces. As the question is local, it is enough to treat them on the half-space  $\mathbb{R}_+^n = \{x_n > 0\}$ , where they are of the form  $P_+ = r^+ P e^+$  for a ps.d.o.  $P$  defined on  $\mathcal{S}'(\mathbb{R}^n)$ , so it suffices to define  $e^+$  on all spaces with  $s$  close to 0 ( $e^+ := e_{\mathbb{R}_+^n}^+$ ,  $r^+ := r_{\mathbb{R}_+^n}^+$ ). However, setting  $e^+ u = \pi(\chi, v)$  when  $r^+ v = u$  and  $\chi$  denotes the characteristic function of  $\mathbb{R}_+^n$ , it follows from (4.4) that  $\pi(\chi, v)$  at most depends on  $v$  in the null set  $\{x_n = 0\}$ . But since the spaces in the next result only contain trivial distributions supported in this hyper-plane, this suffices for a space-independent definition of  $e^+ u$  when  $u$  belongs to these spaces.

**Proposition 4.5.** *The characteristic function  $\chi$  of  $\mathbb{R}_+^n$  yields a bounded map*

$$\pi(\chi, \cdot) : E_{p,q}^s(\mathbb{R}^n) \rightarrow E_{p,q}^s(\mathbb{R}^n), \quad (4.17)$$

*for Besov and Lizorkin–Triebel spaces with  $\frac{1}{p} - 1 + (n-1)(\frac{1}{p} - 1)_+ < s < \frac{1}{p}$ .*

The  $F_{p,q}^s$ -part of this will be based on a similar result of J. Franke [Fra86a, Cor. 3.4.6]. In principle Franke analysed another product as he estimated  $\chi v$  for  $\text{supp } v$  compact and extended by continuity to  $F_{p,q}^s$  (for  $q = \infty$  using Fatou's lemma). But the full treatment of  $P_+$  in  $B_{p,q}^s$  and  $F_{p,q}^s$ -spaces is also based on the splitting of  $\pi$  in (4.8), so it is important that Franke's product equals  $\pi(\chi, v)$ . This was exploited in [Joh96], albeit without details, so it is natural to take the opportunity to return to this point during the

*Proof of Proposition 4.5.* In view of (4.8) it suffices for  $B_{p,q}^s$  to show bounds

$$\|\pi_m(\chi, u)|B_{p,q}^s\| \leq C \|u|B_{p,q}^s\| \quad \text{for } m = 1, 2, 3. \quad (4.18)$$

Using Remark 4.3, this holds for  $m = 1$  for every  $s$  because  $\chi \in L_\infty$ . And  $L_\infty \subset B_{\infty,\infty}^0$ , so for  $m = 2$  it holds for  $s > (\frac{n}{p} - n)_+$ , while for  $m = 3$  it does so for  $s < 0$ . The last two restrictions on  $s$  will be relaxed using the anisotropic structure of  $\chi$ .

For brevity  $u_k := \Phi_k(D)u$ ,  $u^k := \Psi_k(D)u$  etc. Now  $\pi_3(\chi, u) = \sum_{k \geq 2} \chi_k u^{k-2}$ . If  $H$  is the Heaviside function,  $\chi(x) = 1(x') \otimes H(x_n)$  and

$$\chi_k = c \mathcal{F}^{-1}(\Phi_k(\xi) \delta_0(\xi') \otimes \hat{H}(\xi_n)) = 1(x') \otimes \mathcal{F}_{\xi_n \rightarrow x_n}^{-1}(\Phi_k(0, \xi_n) \hat{H}). \quad (4.19)$$

For the second factor, note that  $2^k \hat{H}(2^k \xi_n) = \hat{H}(\xi_n)$  since  $H$  is homogeneous of degree zero, so

$$H_k(x_n) = \mathcal{F}^{-1}(\Phi_1(0, 2^{-k} \cdot) \hat{H})(x_n) = 2^k \mathcal{F}^{-1}(\Phi_1(0, \cdot) \hat{H}(2^k \cdot))(2^k x_n) = H_1(2^k x_n). \quad (4.20)$$

Here  $H_k$  refers to the decomposition  $1 = \sum \Phi_j(0, \xi_n)$  on  $\mathbb{R}$ . For  $k \geq 1$  this gives

$$\|H_k|L_p(\mathbb{R})\| = 2^{-(k-1)/p} \|H_1|L_p(\mathbb{R})\| < \infty. \quad (4.21)$$

Indeed,  $\Phi_1(0, \cdot) \hat{H} \in \mathcal{S}(\mathbb{R})$  because  $\mathcal{F}H = \frac{-i}{\tau} \mathcal{F}(\partial_t H(t)) = \frac{1}{i\tau}$  for  $\tau \neq 0$ ; hence  $H_1 \in L_p$ . Note that  $\tilde{H} := H - H_0$ , by (4.21) and Lemma 2.5, is in  $B_{p,\infty}^{1/p}(\mathbb{R})$  for  $0 < p \leq \infty$ .

To handle the factor  $1(x')$  in (4.19), there is a mixed-norm estimate

$$\|\chi_k u^{k-2}|L_p\|^p \leq \int (\sup_{t \in \mathbb{R}} |u^{k-2}(x', t)|)^p dx' \|H_k|L_p(\mathbb{R})\|^p \quad (4.22)$$

so that  $s - \frac{1}{p} < 0$  in view of the summation lemma (2.14) yields

$$\begin{aligned} \sum_{k \geq 1} 2^{skq} \|\chi_k u^{k-2}\|_p^q &\leq c \sum_{k \geq 1} 2^{(s-\frac{1}{p})kq} \left( \sum_{0 \leq l \leq k} \|u_l|L_p(L_\infty)\|^{\min(1,p)} \right)^{\frac{q}{\min(1,p)}} \|H_1\|_p^q \\ &\leq c \|H_1\|_p^q \sum_{k \geq 0} 2^{(s-\frac{1}{p})kq} \|u_k|L_p(L_\infty)\|^q \\ &\leq c \|\tilde{H}|B_{p,\infty}^{\frac{1}{p}}\|^q \|u|B_{p,q}^s\|^q. \end{aligned} \quad (4.23)$$

Indeed, the last step follows from the Nikol'skiĭ–Plancherel–Polya inequality, cf Lem. 2.6 ff, when this is used in the  $x_n$ -variable (for fixed  $x'$  the Paley–Wiener–Schwartz Theorem gives that  $u(x', \cdot)$  has its spectrum in the region  $|\xi_n| \leq 2^{k+1}$ ). By the dyadic corona criterion, cf Lemma 2.5, this proves  $\pi_3(\chi, u) \in B_{p,q}^s$ , hence the case  $m = 3$  for  $s < \frac{1}{p}$ .

For  $m = 2$  only  $\frac{1}{p} - 1 < s \leq 0$  remains; this implies  $1 < p \leq \infty$ . It can be assumed that  $u_0 = 0$ , for  $u$  may be replaced by  $u - u_0 - u_1$  because  $\chi \in L_\infty$  implies that  $\pi_2(\chi, u_0 + u_1)$  belongs to  $\bigcap_{l \geq 0} B_{p,q}^l$  by Lemma 2.5. Then  $\pi_2(\chi, u)$  is split in three contributions, with details given for  $\sum \chi_k u_k$  (terms with  $\chi_k u_{k-1}$  and  $\chi_{k-1} u_k$

are treated analogously). In the following it is convenient to replace  $(u_j)$  temporarily by  $(0, \dots, 0, u_N, \dots, u_{N+M}, 0, \dots)$ , in which the entries are also called  $u_j$  for simplicity. In this way the below series trivially converge.

Note that the Nikol'skiĭ–Plancherel–Polya inequality used in  $x_n$  yields

$$\|\Phi_j(D) \sum_{k \geq j-1} \chi_k u_k\|_p \leq c \sum_{k \geq j-1} \|\check{\Phi}_j * (\chi_k u_k) |_{L_{p,x'}(L_1, x_n)}\| 2^{j(1-\frac{1}{p})}. \quad (4.24)$$

In this mixed-norm expression, Fubini's theorem gives for  $k \geq 1$

$$\int |\check{\Phi}_j * (\chi_k u_k)(x', x_n)| dx_n \leq \|H_k\|_1 \iint |\check{\Phi}_j(x' - y', x_n)| dx_n \sup_{t \in \mathbb{R}} |u_k(y', t)| dy'. \quad (4.25)$$

Reading this as a convolution on  $\mathbb{R}^{n-1}$ , the usual  $L_p$ -estimate leads to

$$\|\check{\Phi}_j * (\chi_k u_k) |_{L_p(L_1)}\| \leq \|H_k\|_1 \|\check{\Phi}_j\|_1 \|u_k |_{L_p(L_\infty)}\|. \quad (4.26)$$

In view of (4.24) and (2.14) this gives, since  $s+1-\frac{1}{p} > 0$  and  $\text{supp } \mathcal{F}(\chi_k u_k)$  is disjoint from  $\text{supp } \Phi_j$  unless  $k > j-2$  (and since  $u_0 = 0$ ),

$$\begin{aligned} \sum_{j \geq 0} 2^{sjq} \|\check{\Phi}_j * \sum_{k \geq 0} \chi_k u_k\|_p^q &\leq c \sum_{j \geq 0} 2^{(s+1-\frac{1}{p})jq} \left( \sum_{k \geq j-1} \|H_k\|_1 \|u_k |_{L_p(L_\infty)}\| \right)^q \\ &\leq c' \sum_{j \geq 0} 2^{(s+1-\frac{1}{p})jq} \|H_j\|_1^q \|u_j |_{L_p(L_\infty)}\|^q \\ &\leq c' \|\tilde{H} |_{B_{1,\infty}^1(\mathbb{R})}\|^q \sum_{j \geq 0} 2^{sjq} \|u_j\|_p^q < \infty. \end{aligned} \quad (4.27)$$

For  $q < \infty$  the right hand side tends to 0 for  $N \rightarrow \infty$ , so the  $\pi_2$ -series is fundamental in  $B_{p,q}^s$ . There is also convergence for  $q = \infty$ , since  $u \in B_{p,1}^{s-\varepsilon}$  for some  $\varepsilon > 0$  such that  $s - \varepsilon + 1 - \frac{1}{p} > 0$ . The above estimate then also applies to the original  $(u_j)$ , which yields (4.18) for  $m = 2$ .

To cover the  $F_{p,q}^s$ -case, note the continuity  $B_{p,1}^{s+\varepsilon} \xrightarrow{\pi(\chi, \cdot)} B_{p,1}^{s+\varepsilon} \hookrightarrow F_{p,q}^s$  for  $p < \infty$  and sufficiently small  $\varepsilon > 0$ . If Franke's multiplication by  $\chi$  is denoted  $M_\chi$ , it follows that  $B_{p,1}^{s+\varepsilon} \xrightarrow{M_\chi} F_{p,q}^s$  is continuous. Since  $\mathcal{F}^{-1}C_0^\infty$  is dense in  $B_{p,1}^{s+\varepsilon}$  and  $M_\chi$  extends the pointwise product on  $\mathcal{F}^{-1}C_0^\infty$  by  $\chi$ , it follows that  $M_\chi$  coincides with  $\pi(\chi, \cdot)$  for all Besov spaces with  $(s, p, q)$  as in the theorem, if  $p < \infty$ . But then they coincide on all the  $F_{p,q}^s$  spaces, so  $\pi(\chi, \cdot)$  is bounded on  $F_{p,q}^s$  as claimed.  $\square$

*Remark 4.6.* The above direct treatment of the Besov spaces should be of some interest in itself, in view of the mixed-norm estimates that allow a concise proof of all cases.

## 5. PRODUCT TYPE OPERATORS

A basic class of non-linear operators and their parilinearisations can now be formally introduced:

**Definition 5.1.** Operators of *product type*  $(d_0, d_1, d_2)$  on an open set  $\Omega \subset \mathbb{R}^n$  are maps (or finite sums of maps) of the form

$$(v, w) \mapsto P_2 \pi_\Omega(P_0 v, P_1 w), \quad (5.1)$$

for linear partial differential operators  $P_j$  of order  $d_j$ ,  $j = 0, 1, 2$ , with constant coefficients. The quadratic map  $u \mapsto P_2 \pi_\Omega(P_0 u, P_1 u)$  is also said to be of product type.

Although (5.1) often just amounts to  $P_2(P_0 v \cdot P_1 w)$ , it is in general essential to use  $\pi_\Omega$  from (4.5) in this definition, because the product cannot always be reduced to one of the forms in (4.2)–(4.3). In Section 7 below the notion of product type operators will be extended to certain maps between vector bundles.

The case with  $P_2 = I$  is throughout referred to as an operator of type  $(d_0, d_1)$ . Generally  $d_0, d_1, d_2$  appear in the same order as the  $P_j$  are applied.

If for simplicity  $P_2 = I$  is considered, the operator  $\pi_\Omega(P_0 u, P_1 u)$  may of course be viewed as a homogeneous second order polynomial  $p(z_1, \dots, z_N)$  composed with a jet  $J_k u = (D^\alpha u)_{|\alpha| \leq k}$ ,  $k = \max(d_0, d_1)$ . But in general this jet description is too rigid, for a given operator of product type with  $P_2 = I$  may be the restriction of one with  $P_2 \neq I$ , cf Example 5.2. And conversely  $P_2 \pi_\Omega(P_0 u, P_1 u)$  may extend another one of the type in (5.1).

These differences lie not only in the various expressions such operators can be shown to have, but also in the parameter domains they *may* be given. Consider eg

$$u \mapsto u \cdot \partial_1 u, \quad u \mapsto \frac{1}{2} \partial_1(u^2). \quad (5.2)$$

The latter coincides with the former at least for  $u \in C^\infty$ . By Hölder's inequality,  $\partial_1(u^2)$  is a bounded bilinear map  $L_4(\mathbb{R}^n) \rightarrow H^{-1}(\mathbb{R}^n)$ , so its natural parameter domain contains  $(s, p) = (0, 4)$ . But it is not easy to make sense of  $u \partial_1 u$ , as a map  $L_4 \rightarrow H^{-1}$ ; even with the product  $\pi$  it is problematic, for by (4.10b) this is not well defined on  $L_4 \oplus H_4^{-1}$ . Hence it seems best (in analogy with minimal and maximal differential operators in  $L_2(\Omega)$ ) to treat the expressions in (5.2) as two different operators, with different parameter domains.

More general classifications of non-linear operators are available in the literature; the reader may consult eg [Bon81, Sect. 5] and [Yam88, § 2]. But as discussed in the introduction, the product type operators defined above are adequate for fixing ideas and for important applications.

**Example 5.2.** For a useful commutation of differentiations to the left of the point-wise product, consider as in Section 6 below the ‘von Karman bracket’:

$$[v, w] := D_1^2 v D_2^2 w + D_2^2 v D_1^2 w - 2 D_{12}^2 v D_{12}^2 w. \quad (5.3)$$

Introducing the expression

$$B(v, w) = D_{12}^2(D_1 v D_2 w + D_2 v D_1 w) - D_1^2(D_2 v D_2 w) - D_2^2(D_1 v D_1 w), \quad (5.4)$$

then  $B(v, w) = [v, w]$  whenever  $v$  and  $w$  are regular enough to justify application of Leibniz' rule. Clearly  $B(\cdot, \cdot)$  is a case with  $P_2 \neq I$ .

**Definition 5.3.** For each choice of  $\Psi_k$  in (2.4), the *exact parilinearisation*  $L_u$  of  $Q(u) = P_2\pi(P_0u, P_1u)$  on  $\mathbb{R}^n$  is defined as follows,

$$L_u g = -P_2\pi_1(P_0u, P_1g) - P_2\pi_2(P_0u, P_1g) - P_2\pi_3(P_0g, P_1u). \quad (5.5)$$

For  $\Omega \subset \mathbb{R}^n$  and a universal extension operator  $\ell_\Omega$ , cf (1.14), the composite  $g \mapsto r_\Omega L_U(\ell_\Omega g)$  with  $U = \ell_\Omega u$  is also referred to as the exact parilinearisation; it is written  $L_u$  for brevity.

The rationale is that  $L_u g$  has circa the same regularity as  $g$  (contrary to the case of linearisations that are not moderate). Cf Theorem 5.7 below.

Conceptually, Definition 5.3 invokes an interchange of the maps  $\ell_\Omega$  and  $P_j$ , compared to (5.1), where  $P_0$  and  $P_1$  are applied before the implicit extensions to  $\mathbb{R}^n$  in  $\pi_\Omega$ ; cf (4.5). The advantage is that  $L_u g$  then has the structure of a composite map  $r_\Omega \circ P_u \circ \ell_\Omega(g)$  for a certain pseudo-differential operator  $P_u$  of type 1, 1; cf Theorem 5.15 below.

However, as justification  $P_j \ell_\Omega v = \ell_\Omega P_j v$  in  $\Omega$ , whence the localisation property in (4.4) implies that  $-L_u u$  gives back the original product type operator:

**Lemma 5.4.** *Let  $u$  belong to a Besov or Lizorkin–Triebel space  $E_{p,q}^s(\overline{\Omega})$  such that the parameters  $(s - d_j, p, q)_{j=0,1}$  fulfil (4.10)–(4.11) and  $s > \max(d_0, d_1)$ . Then*

$$P_2\pi_\Omega(P_0u, P_1u) = -L_u(u). \quad (5.6)$$

*This holds for any choice of  $\ell_\Omega$  and  $\Psi_k$  (or  $\Phi_k$ ) in the definition of  $L_u$ .*

*Proof.* Let  $P_2 = I$  for simplicity. Theorem 4.2 gives that the parameters  $(s - d_j, p, q)_{j=0,1}$  belong to the parameter domain of  $\pi$  on  $\mathbb{R}^n$ , so it holds for all  $v, w \in E_{p,q}^s(\overline{\Omega})$  that

$$\pi_\Omega(P_0v, P_1w) = r_\Omega\pi(P_0\ell_\Omega v, P_1\ell_\Omega w) = r_\Omega \lim_{k \rightarrow \infty} (\psi_k(D)P_0\ell_\Omega v) \cdot (\psi_k(D)P_1\ell_\Omega w). \quad (5.7)$$

Indeed,  $\pi(P_0\ell_\Omega v, P_1\ell_\Omega w)$  is defined, and  $r_\Omega$  commutes with the limit by its  $\mathcal{D}'$ -continuity, whilst the  $P_j\ell_\Omega v$  restrict to  $P_jv$ , so the product  $\pi_\Omega(P_0, P_1w)$  exists and (5.7) holds.

The choice of  $\ell_\Omega$  is inconsequential for  $r_\Omega\pi(P_0\ell_\Omega v, P_1\ell_\Omega w)$ , since the left hand side of (5.7) does not depend on this; similarly one can take  $\psi_k = \Psi_k$  (the formal definition of  $\pi_\Omega$  in (4.5) is essential here). Now (5.6) follows upon insertion of  $v = w = u$ , for by (4.8) ff and the formula  $\Psi_k = \Phi_0 + \dots + \Phi_k$ , the right hand side of (5.7) then equals the formula for  $-L_u(u)$  in (5.5), since the  $\pi_j$ -series converge by the assumption on  $(s, p, q)$  and the remarks following (4.8).  $\square$

The above introduction of parilinearisation is not the only possible, but the intention here is to make the relation to the ‘pointwise’ product on  $\Omega$  clear.

**5.1. Estimates of product type operators.** For a general product type operator  $B(\cdot, \cdot) := \pi(P_0\cdot, P_1\cdot)$  a large collection of boundedness properties now follows from the theory reviewed in Section 4.1–4.2. Indeed, using Theorem 4.2 it is clear

that  $\pi(P_0, P_1)$  is bounded from  $E_{p_0, q_0}^{s_0} \oplus E_{p_1, q_1}^{s_1}$  to some Besov or Lizorkin–Triebel space provided

$$s_0 + s_1 > d_0 + d_1 + \left(\frac{n}{p_0} + \frac{n}{p_1} - n\right)_+. \quad (5.8)$$

The *standard* domain  $\mathbb{D}(B)$  of the bilinear operator  $B$  is the set of (pairs of triples of) parameters  $(s_j, p_j, q_j)_{j=0,1}$  that satisfy this inequality. Since it works equally well for the *BBB*- and *FFF*-cases, the notation is the same in the two cases.

For the map  $Q(u) := B(u, u)$  the parameter domain  $\mathbb{D}(Q)$  derived from (5.8) is termed the *quadratic standard domain* of  $Q$  (or of  $B$ ). For this domain one has the next result on the *direct* regularity properties of product type non-linearities.

**Proposition 5.5.** *Let  $B(v, w)$  be an operator of product type  $(d_0, d_1, d_2)$  with  $d_0 \leq d_1$ . The quadratic standard domain  $\mathbb{D}(Q)$  consists of the  $(s, p, q)$  fulfilling*

$$s > \frac{d_0 + d_1}{2} + \left(\frac{n}{p} - \frac{n}{2}\right)_+, \quad (5.9)$$

and for each such  $(s, p, q)$  the non-linear operator  $Q$  is bounded

$$Q: B_{p,q}^s \rightarrow B_{p,q}^{s-\sigma(s,p,q)} \quad (5.10)$$

when  $\sigma(s, p, q)$ , for some  $\varepsilon > 0$ , is taken equal to

$$\sigma(s, p, q) = d_2 + d_1 + \left(\frac{n}{p} + d_0 - s\right)_+ + \varepsilon \llbracket \frac{n}{p} + d_0 = s \rrbracket \llbracket q > 1 \rrbracket. \quad (5.11)$$

Similar results hold for  $F_{p,q}^s$  provided  $\llbracket q > 1 \rrbracket$  is replaced by  $\llbracket p > 1 \rrbracket$ .

Analogous results for open sets  $\Omega \subset \mathbb{R}^n$  can be derived from Proposition 4.1. Details on this are left out for simplicity, and so is the proof, for it follows from the below Theorem 5.7 by application of  $L_u$  to  $u$ , cf Lemma 5.4 (note that (5.9) implies  $(s - d_1) + (s - d_0) > 0$ , thence  $s > \max(d_0, d_1)$ ).

*Remark 5.6.* By (5.9) the quadratic domain  $\mathbb{D}(Q)$  only depends on the orders via the mean  $(d_0 + d_1)/2$ . The correction  $\frac{n}{p} - \frac{n}{2}$  occurring for  $p < 2$  is independent of  $d_0$  and  $d_1$ ; cf Figure 2.

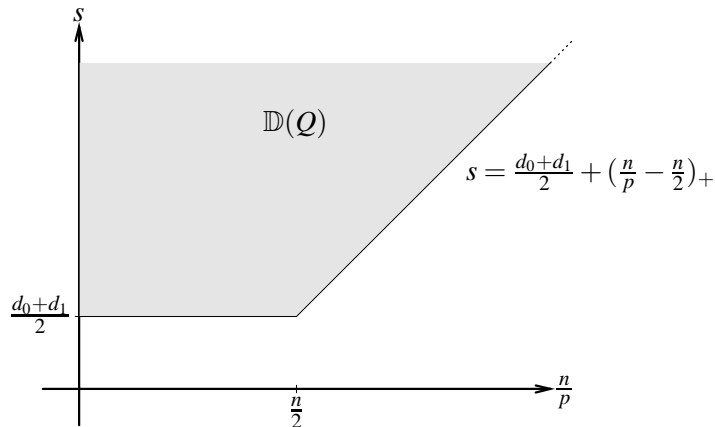


FIGURE 2. The quadratic standard domain  $\mathbb{D}(Q)$



**5.2. Moderate linearisations of product type operators.** The properties of exact parilinearisations of product type operators will now be derived. This will in two ways give better results than the usual linearisation theory in, say [Bon81] and [MC97, Thm. 16.3]: first of all, the  $\pi_2$ -terms are incorporated into  $L_u$ , which is useful since they need not be regularising in the context here. Secondly, the family  $L_u$  is obtained for  $u$  running through the (large) set  $\bigcup B_{p,q}^s$ , and it is only in the quadratic standard domain, where  $u$  is regular enough to make  $-L_u u = Q(u)$  a meaningful formula, that  $L_u$  is a linearisation of  $Q$ .

**Theorem 5.7** (The Exact Parilinearisation Theorem). *Let  $B$  be of product type  $(d_0, d_1, d_2)$  with  $d_0 \leq d_1$  as in Definition 5.1; and let  $\ell_\Omega$  be a universal extension from  $\Omega$  to  $\mathbb{R}^n$ .*

*When  $u \in B_{p_0, q_0}^{s_0}(\overline{\Omega})$  for some arbitrary  $(s_0, p_0, q_0)$ , then the exact parilinearisation in Definition 5.3 yields a linear operator  $L_u$  with parameter domain  $\mathbb{D}(L_u)$  given by*

$$s > d_0 + d_1 - s_0 + \left(\frac{n}{p} + \frac{n}{p_0} - n\right)_+. \quad (5.12)$$

*For  $\varepsilon > 0$  the operator  $L_u$  is of order  $\omega$  as follows,*

$$L_u: B_{p,q}^s(\overline{\Omega}) \rightarrow B_{p,q}^{s-\omega}(\overline{\Omega}) \quad \text{for } (s, p, q) \in \mathbb{D}(L_u), \quad (5.13)$$

$$\omega = d_2 + d_1 + \left(\frac{n}{p_0} - s_0 + d_0\right)_+ + \varepsilon \llbracket \frac{n}{p_0} - s_0 + d_0 = 0 \rrbracket \llbracket q_0 > 1 \rrbracket, \quad (5.14)$$

*In particular, when  $Q(u) := B(u, u)$  and  $(s_0, p_0, q_0) \in \mathbb{D}(Q)$ , cf (5.9), then  $L_u$  is a moderate linearisation of  $Q$ . Corresponding results hold for Lizorkin–Triebel spaces when  $u \in F_{p_0, q_0}^{s_0}(\overline{\Omega})$ , provided the factor  $\llbracket q_0 > 1 \rrbracket$  in (5.14) is replaced by  $\llbracket p_0 > 1 \rrbracket$ .*

**Remark 5.8.** Clearly  $\omega$  is independent of  $(s, p, q)$ ; because it formally equals  $\sigma(s_0, p_0, q_0)$ , it can be said that, for a product type operator, the parilinearisation  $L_u$  inherits the order of the non-linear operator  $Q(u)$  on the space  $E_{p_0, q_0}^{s_0} \ni u$ .

*Proof.* Since the nature of the proof is well known, the formulation will be brief and based on the estimates recalled in Remark 4.3.

In the following  $(s_1, p_1, q_1)$  is arbitrary in  $\mathbb{D}(L_u)$ , ie together with the given  $(s_0, p_0, q_0)$  it fulfils (5.8). It is therefore seen from Remark 4.3 and the Sobolev embeddings that, with  $p_2$  and  $q_2$  as in Remark 4.3,

$$\begin{aligned} \pi_2(P_0 \ell_\Omega \cdot, P_1 \ell_\Omega \cdot): B_{p_0, q_0}^{s_0}(\overline{\Omega}) \oplus B_{p_1, q_1}^{s_1}(\overline{\Omega}) &\rightarrow B_{p_2, q_2}^{s_0 - d_0 + s_1 - d_1} \\ &\hookrightarrow B_{p_1, q_1}^{s_1 - d_1 - (\frac{n}{p_0} - s_0 + d_0)}. \end{aligned} \quad (5.15)$$

The  $\pi_1$ -term in  $L_u$  is straightforward to treat for  $s_0 - d_0 < \frac{n}{p_0}$ : in this case the Sobolev embedding  $B_{p_0, q_0}^{s_0 - d_0} \hookrightarrow B_{\infty, \infty}^{s_0 - d_0 - \frac{n}{p_0}}$  goes into a space with negative smoothness index, so the estimate (4.13) gives, for  $\varepsilon_0 = 0$ ,

$$\pi_1(P_0 \ell_\Omega \cdot, P_1 \ell_\Omega \cdot): B_{p_0, q_0}^{s_0}(\overline{\Omega}) \oplus B_{p_1, q_1}^{s_1}(\overline{\Omega}) \rightarrow B_{p_1, q_1}^{s_1 - d_1 - (\frac{n}{p_0} - s_0 + d_0)_+ - \varepsilon_0}. \quad (5.16)$$

In the same manner one has, since  $u$  appears in the second entry of  $\pi_3$ , that for  $s_0 - d_1 < \frac{n}{p_0}$  and  $\varepsilon_1 = 0$ ,

$$\pi_3(P_0 \ell_{\Omega}, P_1 \ell_{\Omega}) : B_{p_1, q_1}^{s_1}(\overline{\Omega}) \oplus B_{p_0, q_0}^{s_0}(\overline{\Omega}) \rightarrow B_{p_1, q_1}^{s_1 - d_0 - (\frac{n}{p_0} - s_0 + d_1)_+ - \varepsilon_1}. \quad (5.17)$$

For  $s_0 - d_0 > \frac{n}{p_0}$  the estimate (4.12) and  $B_{p_0, q_0}^{s_0 - d_0} \hookrightarrow L_{\infty}$  clearly yields the conclusion in (5.16) with  $\varepsilon_0 = 0$ . The term with  $\pi_3$  may be treated analogously for  $s_0 - d_1 > \frac{n}{p_0}$ , leading to (5.17) once again. For  $s_0 - d_j = \frac{n}{p_0}$  one can use (5.16) and (5.17) at the expense of some  $\varepsilon_j > 0$ , eg fulfilling  $0 < \varepsilon_1 < d_1 - d_0$ , or  $\varepsilon_0 = \varepsilon_1$  if  $d_1 = d_0$ . This is unless  $q_0 \leq 1$  for then the embedding into  $L_{\infty}$  applies.

Comparing the three estimates (incl. the  $\varepsilon$ -modifications), (5.15) is the same as (5.16), except when  $\frac{n}{p_0} - s_0 + d_0 \leq 0$ , but in this case  $B_{p_1, q_1}^{s_1 - d_1}$  or  $B_{p_1, q_1}^{s_1 - d_1 - \varepsilon_0}$  in (5.16) clearly contains the space on the right hand side of (5.15). Similarly the co-domain of (5.17) equals the last space in (5.15), except for  $\frac{n}{p_0} - s_0 + d_1 \leq 0$ , but then the assumption that  $d_0 \leq d_1$  yields that also  $\frac{n}{p_0} - s_0 + d_0 \leq 0$  so that there is an embedding into the corresponding space in (5.16). Regardless of whether  $(\frac{n}{p_0} - s_0 + d_j)_+$  equals 0 for none, one or both  $j$  in  $\{0, 1\}$ , it follows that  $L_u$  is a bounded linear operator

$$L_u : B_{p_1, q_1}^{s_1} \rightarrow B_{p_1, q_1}^{s_1 - \omega}, \quad (5.18)$$

when  $\omega$  is as in (5.14) and  $(s_1, p_1, q_1)$  fulfils (5.8).

In the Lizorkin–Triebel case the above argument works with minor modifications. For one thing the Sobolev embedding  $F_{p_2, t}^{s_0 - d_0 + s_1 - d_1} \hookrightarrow F_{p_1, q_1}^{s_1 - d_1 - (\frac{n}{p_0} - s_0 + d_0)}$  and (4.15) give an analogue of (5.15).

Secondly, for  $s_0 - d_0 < \frac{n}{p_0}$ , it is easy to see from the dyadic corona criterion and the summation lemma (in analogy with the proof of Lemma 2.7) that if  $r < 0$ ,

$$\pi_1(\cdot, \cdot) : B_{\infty, \infty}^r \oplus F_{p_1, q_1}^{s_1} \rightarrow F_{p_1, q_1}^{s_1 + r}. \quad (5.19)$$

Combining this with  $F_{p_0, q_0}^{s_0 - d_0} \hookrightarrow B_{\infty, \infty}^{s_0 - d_0 - \frac{n}{p_0}}$ , formula (5.16) is carried over to the Lizorkin–Triebel case. Otherwise one may proceed as in the Besov case, noting that  $F_{p, q}^{n/p} \hookrightarrow L_{\infty}$  when  $p \leq 1$ .  $\square$

To shed light on (5.12), one could consider an elliptic problem  $\{A, T\}$ , say with  $A$  of order  $2m$ ,  $T$  of class  $m$  and a solution  $u \in H^m(\overline{\Omega})$ , with  $(m, 2) \in \mathbb{D}(Q)$ , of

$$Au + Q(u) = f \quad \text{in } \Omega \quad (5.20)$$

$$Tu = \varphi \quad \text{on } \Gamma. \quad (5.21)$$

According to (5.12),  $\mathbb{D}(L_u)$  then consists of parameters  $(s, p, q)$  with

$$s > \frac{d_0 + d_1}{2} + \left(\frac{n}{p} - \frac{n}{2}\right)_+ - \left(m - \frac{d_0 + d_1}{2}\right), \quad (5.22)$$

so that  $\mathbb{D}(L_u)$  is obtained from the quadratic standard domain  $\mathbb{D}(Q)$  in (5.9) simply by a downward shift given by the last parenthesis, which is positive for  $(m, 2) \in \mathbb{D}(Q)$ . Therefore  $\mathbb{D}(L_u) \supset \mathbb{D}(Q)$ ; by an extension of the argument this is seen to hold also in general when  $(s_0, p_0, q_0) \in \mathbb{D}(Q)$ .

When deriving easy-to-apply criteria for  $A$ -moderacy, for some given linear operator  $A$  of constant order  $d_A$  on a parameter domain  $\mathbb{D}(A)$ , it is clearly a necessary condition that  $d_A > d_2 + \max(d_0, d_1)$ , for both  $\sigma$  and  $\omega$  are  $\geq d_2 + \max(d_0, d_1)$ .

**Corollary 5.9.** *Let  $Q(u)$  be of product type  $(d_0, d_1, d_2)$  with  $d_0 \leq d_1$ . When  $d_A > d_2 + d_1$ , then  $Q$  is  $A$ -moderate on every  $E_{p,q}^s$  in  $\mathbb{D}(A) \cap \mathbb{D}(Q)$  if  $d_1 - d_0 \geq n$ , or else on the  $E_{p,q}^s$  in  $\mathbb{D}(A) \cap \mathbb{D}(Q)$  fulfilling*

$$s > \frac{n}{p} - d_A + d_0 + d_1 + d_2. \quad (5.23)$$

*The exact parilinearisation  $L_u$  is  $A$ -moderate on  $\mathbb{D}(L_u)$  when  $Q$  is  $A$ -moderate on the space  $E_{p_0, q_0}^{s_0} \ni u$ .*

*Proof.* Given (5.23) one has  $d_A - d_2 - d_1 > (\frac{n}{p} - s + d_0)_+ \geq 0$ . So by taking  $\varepsilon \in ]0, d_A - d_2 - d_1[$ , clearly this gives  $d_A > \sigma$  so that  $Q$  is  $A$ -moderate on  $E_{p,q}^s$ . However, if  $d_1 - d_0 \geq n$  it is easy to see, both for  $p < 2$  and  $p \geq 2$ , that every  $(s, p, q)$  fulfills

$$\frac{1}{2}(d_0 + d_1) + (\frac{n}{p} - \frac{n}{2})_+ \geq \frac{n}{p} + d_0. \quad (5.24)$$

Consequently  $s > \frac{n}{p} + d_0$ , so  $\sigma = d_2 + d_1$ . Hence  $Q$  is  $A$ -moderate on the entire domain  $\mathbb{D}(A) \cap \mathbb{D}(Q)$  in this case.

The statement on  $L_u$  follows since  $\omega$  equals  $\sigma$  on the space containing the linearisation point  $u$ .  $\square$

In cases with  $d_1 - d_0 < n$ , there always is a part of the quadratic standard domain  $\mathbb{D}(Q)$  where (5.23) must be imposed. Indeed, the last two terms in (5.11) contributes to the value of  $\sigma$  in the *slanted slice* of  $\mathbb{D}(Q)$  given by

$$\frac{1}{2}(d_0 + d_1) + (\frac{n}{p} - \frac{n}{2})_+ < s \leq \frac{n}{p} + d_0. \quad (5.25)$$

For  $d_1 - d_0 < n$  any  $p < 2$  leads to solutions  $(s, p)$  of these inequalities, so the slice in (5.25) is non-empty. Because  $\sigma > d_2 + d_1$  in the slice,  $A$ -moderacy is obtained only where  $d_A > \sigma$ , ie where (5.23) holds. Note, however, that  $L_u$  by the formulae for  $\sigma$  and  $\omega$  is born to be  $A$ -moderate on the entire domain  $\mathbb{D}(L_u)$ , if only  $Q$  is so on a space containing  $u$ .

**Remark 5.10.** Concerning the model problem (1.3) and Example 3.1, where  $d_0 = 0$ ,  $d_1 = 1$  and  $d_A = 2$ , the above (5.9) leads to the quadratic standard domains in (1.27) and (3.15). Notice that the more important domains  $\mathbb{D}(\mathcal{A}, Q)$  and  $\mathbb{D}(A, \mathcal{N})$  in (1.30) and (3.16) are obtained from the conjunction of (5.9) and (5.23) (the latter is redundant for  $n = 2$  and  $n = 3$ ). Similarly (3.17) follows from (5.12).

**Remark 5.11.** One could compare (1.3) (or the stationary Navier–Stokes problem) with the von Karman problem (cf Section 6). They both fulfil  $d_1 - d_0 \leq 1 < n$ . In the former problem (5.23) is felt, and the quadratic term is only  $\Delta_{\gamma_0}$ -moderate on the part of  $\mathbb{D}(Q) \cap \mathbb{D}_1$  where  $s > \frac{n}{p} - 1$ , by (5.23). (For the Neumann condition, (5.23) gives again  $s > \frac{n}{p} - 1$ , that now should be imposed on the smaller region  $\mathbb{D}(Q) \cap \mathbb{D}_2$  because the boundary condition has class 2.) But in the von Karman problem, (5.23) is not felt, for it is fulfilled on all of the quadratic standard domain of the form  $[\cdot, \cdot]$ , and even after this has been extended to the  $B(\cdot, \cdot)$  of type  $(1, 1, 2)$

given in Example 5.2, it *still* holds that  $\omega < 4 = d_{\Delta^2}$  on all of  $\mathbb{D}(Q)$ . But nevertheless a small portion of  $\mathbb{D}(Q)$  must be disregarded to have  $\Delta^2$ -moderacy, simply because the boundary condition in the Dirichlet realisation of  $\Delta^2$  is felt; cf Figure 3 below. In view of this, it seems pointless to generalise beyond Corollary 5.9.

**5.3. Boundedness in a borderline case.** In the cases given by equality in (4.11) it is more demanding to estimate  $L_u$ . For later reference a first result on such extensions of  $\mathbb{D}(L_u)$  is sketched. It adopts techniques from a joint work with W. Farkas and W. Sickel [FJS00], in which approximation spaces  $A_{p,q}^s$  (that go back to S. M. Nikol'skiĭ) were useful for the borderline investigations.

Recall that  $A_{p,q}^s(\mathbb{R}^n)$  for  $s \geq (\frac{n}{p} - n)_+$ ,  $p, q \in ]0, \infty]$  (with  $q \leq 1$  for  $s = \frac{n}{p} - n$ ), consists of the  $u \in \mathcal{S}'(\mathbb{R}^n)$  that have an  $\mathcal{S}'$ -convergent decomposition  $u = \sum_{j=0}^{\infty} v_j$  fulfilling  $\text{supp } \hat{v}_j \subset \{|\xi| \leq 2^{j+1}\}$  for  $v_j \in \mathcal{S}' \cap L_p$  with

$$\left( \sum_{j=0}^{\infty} 2^{sjq} \|v_j\|_{L_p}^q \right)^{\frac{1}{q}} < \infty. \quad (5.26)$$

Then  $\|u\|_{A_{p,q}^s}$  is the infimum of these numbers, over all such decompositions.

The idea of [FJS00] is that, while the dyadic ball criterion cannot yield convergence for  $s = \frac{n}{p} - n$  (at least not for  $q > 1$ ), one can sometimes show directly that such  $\sum v_j$  converges to some  $u$  in  $L_1$  or  $\mathcal{S}'$ ; then the finiteness of the above number gives  $\sum v_j \in A_{p,q}^s$ . For this purpose the next borderline result is recalled from [Joh95, Prop. 2.5].

**Lemma 5.12.** *Let  $0 < q \leq 1 \leq p < \infty$  and let  $\sum_{j=0}^{\infty} u_j$  be such that  $F(q) < \infty$  for  $F(q) = \|(\sum |u_j|^q)^{1/q}\|_p$ . Then  $\sum u_j$  converges in  $L_p$  to a sum  $u$  fulfilling  $\|u\|_{L_p} \leq F(q)$ .*

*Proof.* With  $\sum |u_j(x)|$  as a majorant (since  $F(1) \leq F(q)$ ),  $\|\sum_{j=k}^{\infty} |u_j|\|_{L_p} \xrightarrow[k \rightarrow \infty]{} 0$ . Hence  $\sum u_j$  is a fundamental series in  $L_p$ , and the estimate follows.  $\square$

**Theorem 5.13.** *Let  $B = \pi_{\Omega}(P_0 \cdot, P_1 \cdot)$  with  $d_0 \leq d_1$  and let  $u \in B_{p_0, q_0}^{s_0}(\overline{\Omega})$  be fixed. For  $(s, p, q)$  such that*

$$s_0 + s = d_0 + d_1 + \left(\frac{n}{p_0} + \frac{n}{p} - n\right)_+, \quad \frac{1}{q_2} := \frac{1}{q_0} + \frac{1}{q} \geq 1 \quad (5.27)$$

*the operator  $L_u$  is continuous*

$$L_u : B_{p,q}^s(\overline{\Omega}) \rightarrow B_{p,\infty}^{s-\omega}(\overline{\Omega}), \quad (5.28)$$

*provided, in case  $\frac{1}{p_2} := \frac{1}{p_0} + \frac{1}{p} > 1$ , that  $p_2 \geq q_2$  or  $p \geq 1$  holds. Moreover,  $L_u : F_{p,q}^s(\overline{\Omega}) \rightarrow B_{p,\infty}^{s-\omega}(\overline{\Omega})$  is continuous if  $u \in F_{p_0, q_0}^{s_0}(\overline{\Omega})$ , when  $\llbracket q_0 > 1 \rrbracket$  in (5.14) is replaced by  $\llbracket p_0 > 1 \rrbracket$  (no restrictions for  $p_2 < 1$ ).*

*Proof.* With notation as in the proof of Theorem 5.7, the assumption  $q_2 \leq 1$  gives  $\ell_{q_2} \hookrightarrow \ell_1$ , so for  $p_2 \geq 1$  insertion of  $1 = 2^{s_0 - d_0 + s_1 - d_1}$  into a double application of Hölder's inequality shows that the series defining  $\pi_2(P_0 \ell_{\Omega} \cdot, P_1 \ell_{\Omega} \cdot)$  converges absolutely in  $L_{p_2}$ . There is a Sobolev embedding  $L_{p_2} \hookrightarrow B_{p_1, \infty}^{\tilde{s}}$  for  $\tilde{s} = s_1 - d_1 - (\frac{n}{p_0} - s_0 + d_0)$ , since  $p_1 \geq p_2$ , so the conclusion of (5.15) holds with the modification that the sum-exponent is  $\infty$  in this case.

For  $p_2 < 1$  one uses the Nikol'skiĭ–Plancherel–Polya inequality to estimate  $L_1$ -norms by  $2^{\frac{n}{p_2}-n} = 2^{s_0+s_1-d_0-d_1}$  times corresponding  $L_{p_2}$ -norms, leading to convergence in  $L_1$ . After this convergence has been established, the same estimates also give the strengthened conclusion that, for  $A_{p,q}^s$  as above,

$$\pi_2(P_0\ell_\Omega, P_1\ell_\Omega) : B_{p_0,q_0}^{s_0} \oplus B_{p_1,q_1}^{s_1} \rightarrow A_{p_2,q_2}^{\frac{n}{p_2}-n}. \quad (5.29)$$

By [FJS00, Thm. 6] the conjunction of  $r \geq \max(p_2, q_2)$  and  $o = \infty$  is equivalent to

$$A_{p_2,q_2}^{\frac{n}{p_2}-n} \hookrightarrow B_{r,o}^{\frac{n}{r}-n}. \quad (5.30)$$

Therefore  $\pi_{2\Omega}^{12}(u, \cdot) := r_\Omega \pi_2(P_0\ell_\Omega u, P_1\ell_\Omega \cdot)$  is continuous  $B_{p_1,q_1}^{s_1}(\overline{\Omega}) \rightarrow B_{p_2,\infty}^{\frac{n}{p_2}-n}(\overline{\Omega})$  for  $p_2 \geq q_2$ , hence into  $B_{p_1,\infty}^{\tilde{s}}(\overline{\Omega})$  as desired; for  $p_1 \geq 1$  this is seen directly from the above  $L_1$ -estimate.

Since (5.16) and (5.17) also hold in the present context, and since this implies weaker statements with the sum-exponents equal to  $\infty$  on the right hand sides there,  $L_u$  has the property in (5.18) except that the co-domain should be  $B_{p_1,\infty}^{s_1-\omega}$ .

For the  $F_{p,q}^s$ -spaces the estimates of  $\pi_{2\Omega}^{12}(u, \cdot)$  are derived in the same way, except that the  $\ell_{q_2}$ -norms are calculated pointwisely, before the  $L_{p_2}$ -norms. Indeed, for  $p_2 \geq 1$ , Lemma 5.12 gives (since  $q_2 \leq 1$  in this case) that  $\pi_2(P_0\ell_\Omega, P_1\ell_\Omega)$  maps  $F_{p_0,q_0}^{s_0} \oplus F_{p_1,q_1}^{s_1}$  to  $L_{p_2}$ : for  $p_2 > 1$  this co-domain is embedded via  $F_{p_1,q_1}^{\tilde{s}}$  into  $B_{p_1,\infty}^{\tilde{s}}$ , while  $L_{p_2} \hookrightarrow B_{1,\infty}^0 \hookrightarrow B_{p_1,\infty}^{\tilde{s}}$  for  $p_2 = 1$ .

For  $p_2 < 1$  one finds by the vector-valued Nikol'skiĭ–Plancherel–Polya inequality in Lemma 2.6 that eg (when  $f_k := \Phi_k(D)f$  etc on  $\mathbb{R}^n$ )

$$\left\| \sum_{k=0}^{\infty} |f_k g_k| \right\|_1 \leq c \left\| \left( \sum_{k=0}^{\infty} 2^{k(\frac{n}{p_2}-n)q_2} |f_k g_k|^{q_2} \right)^{\frac{1}{q_2}} \right\|_{p_2} \leq c' \|f\|_{F_{p_0,q_0}^{s_0-d_0}} \|g\|_{F_{p_1,q_1}^{s_1-d_1}}. \quad (5.31)$$

In this way  $\pi_{2\Omega}^{12}(u, \cdot)$  is shown to map  $F_{p_1,q_1}^{s_1}$  into  $L_1(\Omega)$ . Hence into  $B_{p_1,\infty}^{\tilde{s}}(\overline{\Omega})$  for  $p_1 \geq 1$ . In general there is  $p_3 \in ]p_2, p_1[$  ( $p_0 < \infty$ ) and the  $A_{p_3,p_3}^{\frac{n}{p_3}-n}$ -norm of  $\pi_{2\Omega}^{12}(u, v)$  is estimated by an  $L_{p_2}(\ell_{q_2})$ -norm as in the middle of (5.31), for the sum and integral may be exchanged and the estimate realised through Lemma 2.6. By (5.30)–(5.31) this means that  $\pi_{2\Omega}^{12}(u, \cdot)$  maps  $F_{p_1,q_1}^{s_1}$  into  $B_{p_3,\infty}^{\frac{n}{p_3}-n} \hookrightarrow B_{p,\infty}^{\tilde{s}}$  for  $p_2 < 1$ . Comparison with the  $F_{p,q}^s$ -results for the other terms shows that  $L_u : F_{p_1,q_1}^{s_1} \rightarrow B_{p_1,\infty}^{\tilde{s}}$ .  $\square$

The above result suffices for the present paper, but it could probably be sharpened in several ways, perhaps with a consistent use of  $A_{p,q}^s$  as co-domains.

**5.4. Relations to pseudo-differential operators of type 1,1.** For the local regularity improvements later, it is useful to express parilinearisations in terms of pseudo-differential operators with symbols in  $S_{1,1}^d$ . Recall that  $a(x, \xi) \in C^\infty(\mathbb{R}^{2n})$  belongs to  $S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n)$  for  $d \in \mathbb{R}$ , if for all multiindices  $\alpha, \beta$  there is  $c_{\alpha\beta} > 0$  such that for  $x, \xi \in \mathbb{R}^n$ ,

$$|D_x^\beta D_\xi^\alpha a(x, \xi)| \leq c_{\alpha\beta} \langle \xi \rangle^{d-|\alpha|+|\beta|}; \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2}. \quad (5.32)$$

The operator  $a(x, D)\varphi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi) \hat{\varphi}(\xi) d\xi$  obviously induces a bilinear map  $S_{1,1}^d \times \mathcal{S} \rightarrow \mathcal{S}$  that is continuous with respect to the Fréchet topologies. In general  $A := a(x, D) = \text{OP}(a)$  cannot be extended to  $\mathcal{S}'$  by duality, for the adjoint of  $A$  need not be of type 1, 1. However,  $A$  can be defined as a linear operator with domain  $D(A) \subset \mathcal{S}'(\mathbb{R}^n)$  by analogy with (4.1). More precisely  $u \in \mathcal{S}'$  is in  $D(A)$  when the limit

$$a_\psi(x, D)u := \lim_{k \rightarrow \infty} \text{OP}(\psi_k(D_x)a(x, \xi)\psi_k(\xi))u \quad (5.33)$$

exists in  $\mathcal{S}'(\mathbb{R}^n)$  for all  $\psi \in C_0^\infty(\mathbb{R}^n)$  with  $\psi = 1$  in a neighbourhood of the origin, and when moreover  $a_\psi(x, D)u$  is independent of such  $\psi$  so that it makes sense to let  $a(x, D)u = a_\psi(x, D)u$  then.

This definition by so-called vanishing frequency modulation was introduced recently and investigated from several perspectives in [Joh08]. As the symbol on the right-hand side of (5.33) is in  $S^{-\infty}$  the definition means roughly that in  $a(x, D)u$  one should regularise the symbol  $a$  instead of the argument  $u$ ; it clearly gives the integral after (5.32) for  $u \in \mathcal{S}(\mathbb{R}^n)$ .

Previously L. Hörmander determined (up to a limit point) the  $s$  for which  $A$  extends to a continuous map  $H_2^{s+d} \rightarrow H_2^s$ ; cf [Hör88, Hör89] and [Hör97, Ch 9.3]. Eg continuity for all  $s \in \mathbb{R}$  is proved there for  $a(x, \xi)$  satisfying his twisted diagonal condition. However, it was proved in [Joh04, Joh05] that there always are bounded extensions, for  $1 \leq p < \infty$ ,

$$F_{p,1}^d(\mathbb{R}^n) \xrightarrow{a(x,D)} L_p(\mathbb{R}^n), \quad B_{\infty,1}^d(\mathbb{R}^n) \xrightarrow{a(x,D)} L_\infty(\mathbb{R}^n), \quad (5.34)$$

and that, without further knowledge about  $a(x, \xi)$ , this is optimal within the  $B_{p,q}^s$  and  $F_{p,q}^s$  scales for  $p < \infty$ . For  $s > (\frac{n}{p} - n)_+$  there is continuity

$$B_{p,q}^{s+d}(\mathbb{R}^n) \xrightarrow{a(x,D)} B_{p,q}^s(\mathbb{R}^n), \quad F_{p,q}^{s+d}(\mathbb{R}^n) \xrightarrow{a(x,D)} F_{p,r}^s(\mathbb{R}^n) \quad (r \text{ as in (2.12)}). \quad (5.35)$$

This extends to all  $s \in \mathbb{R}$  under the twisted diagonal condition; cf [Joh05, Cor. 6.2]; cf also [Joh08]. The reader may consult [Hör97, Joh08] for various aspects of the theory of operators in  $\text{OP}(S_{1,1}^d)$ .

The just mentioned results will not be directly used here, but they shed light on how difficult it is to determine the domain  $D(A)$ . Nevertheless one has the *pseudo-local* property:

$$\text{sing supp } Au \subset \text{sing supp } u \quad \text{for all } u \in D(A). \quad (5.36)$$

**Theorem 5.14.** *Every pseudo-differential operator  $a(x, D)$  in  $\text{OP}(S_{1,1}^d(\mathbb{R}^n \times \mathbb{R}^n))$  has the property in (5.36).*

This was first proved in [Joh08, Thm. 6.4], to which the reader is referred. The proof given there exploits the definition of type 1, 1-operators given above as well as the Regular Convergence Lemma; cf Lemma 2.1.

The exact parilinearisations turn out to factor through pseudo-differential operators of type 1, 1. This entails that the former are pseudo-local:

**Theorem 5.15.** *Let  $B$  be of product type and  $u \in B_{p_0, q_0}^{s_0}(\overline{\Omega})$  for some arbitrary  $(s_0, p_0, q_0)$ . Then the exact parilinearisation in (5.5) factors through an operator  $P_u \in \text{OP}(S_{1,1}^\omega(\mathbb{R}^n \times \mathbb{R}^n))$  with  $\omega$  as in (5.14). That is, for every  $(s, p, q)$  in  $\mathbb{D}(L_u)$ , cf (5.12), there is a commutative diagram*

$$\begin{array}{ccc} E_{p,q}^s(\overline{\Omega}) & \xrightarrow{\ell_\Omega} & E_{p,q}^s(\mathbb{R}^n) \\ L_u \downarrow & & \downarrow P_u \\ E_{p,q}^{s-\omega}(\overline{\Omega}) & \xleftarrow{r_\Omega} & E_{p,q}^{s-\omega}(\mathbb{R}^n). \end{array} \quad (5.37)$$

Moreover,  $g \mapsto L_u g$  is pseudo-local when  $g \in E_{p,q}^s(\overline{\Omega})$  and  $(s, p, q)$  is in  $\mathbb{D}(L_u)$ .

*Proof.* 1°. By linearity, it suffices to treat  $P_m = D^{\eta_m}$  for  $|\eta_m| = d_m$ , and  $d_0 \leq d_1$ ,  $d_2 = 0$ . Set  $\tilde{u} = \ell_\Omega u$ .

2°. Applying  $L_u$  to  $\ell_\Omega g \in \mathcal{S}$ , it is a composite  $L_u = r_\Omega a(x, D) \ell_\Omega$  for a symbol  $a(x, \xi)$  satisfying (5.32) for  $d = \omega$  with  $\omega$  as in (5.14), namely

$$a(x, \xi) = - \sum_{j=0}^{\infty} (\Psi_{j+1}(D_x) D_x^{\eta_0} \tilde{u}(x) \xi^{\eta_1} + \Psi_{j-2}(D_x) D_x^{\eta_1} \tilde{u}(x) \xi^{\eta_0}) \Phi_j(\xi) \quad (5.38)$$

Indeed, the formula for  $a(x, \xi)$  follows directly from Definition 5.3 and (4.8) once  $a \in S_{1,1}^\omega$  has been verified. To prove that  $P_u = a(x, D)$  is of type 1, 1, note that  $a(x, \xi)$  is  $C^\infty$  since each  $\xi$  is in  $\text{supp } \Phi_j$  for at most two values of  $j$ , and for these  $2^{j-1} \leq |\xi| \leq 2^{j+1}$ , so that  $|D^\alpha(\xi^{\eta_m} \Phi_j(\xi))| \leq c \langle \xi \rangle^{d_m - |\alpha|}$  holds for all  $\alpha$ . Concerning the estimates for  $x \in \mathbb{R}^n$  and  $\xi \in \text{supp } \Phi_j$ , so that  $2^j \leq 2 \langle \xi \rangle$ , note that if  $k = j + 1$  and  $\varepsilon > 0$  is fixed, the convenient short-hand  $\varepsilon' := \varepsilon \llbracket \frac{n}{p_0} - s_0 + d_0 = 0 \rrbracket \llbracket q_0 > 1 \rrbracket$  fulfils  $\varepsilon' \geq 0$  and gives

$$|D_x^\beta \Psi_k(D) D^{\eta_0} \tilde{u}(x)| \leq c \langle \xi \rangle^{|\beta| + (\frac{n}{p_0} - s_0 + d_0)_+ + \varepsilon'}. \quad (5.39)$$

In fact, for  $q_0 \leq 1$  one has  $\ell_q \hookrightarrow \ell_1$ , so the Nikol'skiĭ–Plancherel–Polya inequality yields

$$\begin{aligned} |\Psi_k(D) D^{\beta + \eta_0} \tilde{u}(x)| &\leq c \sum_{l=0}^k 2^{l(s_0 - |\beta + \eta_0|)} \|\Phi_l(D) D^{\beta + \eta_0} \tilde{u}\|_{L_{p_0}} 2^{l(|\beta| + \frac{n}{p_0} - s_0 + d_0)} \\ &\leq c \|u\|_{B_{p_0, q_0}^{s_0}} \|\langle \xi \rangle^{(\frac{n}{p_0} - s_0 + d_0)_+ + |\beta|}\|, \end{aligned} \quad (5.40)$$

for  $q_0 > 1$  Hölder's inequality applies to the first line in (5.40), if  $2^{k(\frac{n}{p_0} - s_0 + d_0)_+ + k|\beta|}$  is taken out in front of the summation (it is less than  $(4 \langle \xi \rangle)^{|\beta| + (\frac{n}{p_0} - s_0 + d_0)_+}$ ); except when  $\frac{n}{p_0} - s_0 + d_0 = 0$ , ie  $\varepsilon' > 0$ , then  $|\beta|$  should just have  $\varepsilon$  added and subtracted. This shows (5.39).

Terms with  $|\Psi_{j-2}(D) D^{\beta + \eta_1} \tilde{u}(x)|$  are treated analogously, in the first line of (5.40) the factor  $2^{l(s_0 - |\beta + \eta_0|)}$  may be estimated by  $2^{l(s_0 - |\beta + \eta_1|)}$  (which is absorbed by the Besov norm on  $u$ ) times  $2^{j(d_1 - d_0)}$ ; the latter, together with the estimate of  $D^\alpha(\xi^{\eta_0} \Phi_j(\xi))$ , gives the estimates in (5.32) also for these terms.

3°. To prove (5.37) also for non-smooth functions, it is noted that there is a linear map  $P_u: E_{p,q}^s(\mathbb{R}^n) \rightarrow E_{p,q}^{s-\omega}(\mathbb{R}^n)$  that is bounded for  $(s, p, q) \in \mathbb{D}(L_u)$ . This

is seen as in the proof of Theorem 5.7, cf (5.18), for one can keep the first entry in the expressions with  $\pi_1, \pi_2$  there equal to  $P_0\tilde{u}$  while the other entry runs through  $P_1(E_{p,q}^s(\mathbb{R}^n))$ , for  $\pi_3$  the first entry is taken in  $P_0(E_{p,q}^s)$  and the second equal to  $P_1\tilde{u}$ . From the definition of  $L_u$  it is then evident that  $L_u = r_\Omega \circ P_u \circ \ell_\Omega$ , hence (5.37) holds.

4°. To show that  $P_u$  from step 3° equals the type 1,1-operator  $a(x,D)$  given by the symbol in step 2°, it remains by (5.33) to be verified that one has the limit relation  $P_u f = \lim_{m \rightarrow \infty} \text{OP}(\psi_m(D_x)a(x,\xi)\psi_m(\xi))f$  for all  $\psi \in C_0^\infty(\mathbb{R}^n)$  with  $\psi = 1$  around 0, whenever  $f \in B_{p,q}^s(\mathbb{R}^n)$  with  $(s,p,q) \in \mathbb{D}(L_u)$ , ie for

$$s_0 - d_0 + s - d_1 > \max(0, \frac{n}{p_0} + \frac{n}{p} - n). \quad (5.41)$$

(If  $f \in F_{p,q}^s$ , then  $f \in B_{p,\infty}^s$  that also fulfils (5.41).) This is tedious but results from consistent use of the techniques that gave boundedness of  $P_u$ .

Indeed, for every  $\psi$  and a (large)  $m$  as above, it is straightforward to see that  $\Phi_j \psi_m = \Phi_j$  and  $\Psi_{j+1} \psi_m = \Psi_{j+1}$  for  $j$  below a certain limit  $J(m)$ , so that the symbol of the approximating operator can be written as follows, when ' indicates summation over  $l = m - j$  in a fixed finite subset of  $\mathbb{Z}$ ,

$$\begin{aligned} \psi_m(D_x)a(x,\xi)\psi_m(\xi) = & - \sum_{j \leq J(m)} (\Psi_{j+1}(D_x)D_x^{\eta_0}\tilde{u}(x)\xi^{\eta_1} + \\ & \Psi_{j-2}(D_x)D_x^{\eta_1}\tilde{u}(x)\xi^{\eta_0})\Phi_j(\xi) \\ & - \sum_l' (\psi_m(D_x)\Psi_{m-l+1}(D_x)D_x^{\eta_0}\tilde{u}(x)\xi^{\eta_1} + \\ & \psi_m(D_x)\Psi_{m-l-2}(D_x)D_x^{\eta_1}\tilde{u}(x)\xi^{\eta_0})\Phi_{m-l}(\xi)\psi_m(\xi). \end{aligned} \quad (5.42)$$

The operator induced by the first sum here converges to  $P_u$  for  $m \rightarrow \infty$ , by (5.38) and the construction of  $P_u$ . Therefore it suffices to show that the primed sum defines an operator  $R_m$  for which  $R_m f \rightarrow 0$  for  $m \rightarrow \infty$ . Fixing  $l$  one has the contribution

$$\begin{aligned} R_{l,m}f = & (\psi_m(D)\Psi_{m-l+1}(D)D^{\eta_0}\tilde{u} \cdot D^{\eta_1} \\ & + \psi_m(D)\Psi_{m-l-2}(D)D^{\eta_1}\tilde{u}(x) \cdot D^{\eta_0})\Phi_{m-l}(D)\psi_m(D)f, \end{aligned} \quad (5.43)$$

the worst part of which is

$$\tilde{R}_{l,m}f = \psi_m(D)(\Phi_{m-l-1} + \Phi_{m-l} + \Phi_{m-l+1})(D)D^{\eta_0}\tilde{u} \cdot D^{\eta_1}\Phi_{m-l}(D)\psi_m(D)f. \quad (5.44)$$

Clearly  $\text{supp } \tilde{R}_{l,m}f$  is contained in  $B(0, c2^m)$ , ie it fulfils the dyadic ball condition in Lemma 2.3. To estimate the quantity  $B$  there, note that in case  $p, p_0 \geq 1$  the family  $\psi_m(D)$  is uniformly bounded in  $L_p$  and  $L_{p_0}$ , so when  $\frac{1}{p_2} = \frac{1}{p_0} + \frac{1}{p}$  and



$\frac{1}{q_2} = \frac{1}{q_0} + \frac{1}{q}$ , then

$$\begin{aligned} \left( \sum_{m=0}^{\infty} 2^{(s_0+s-(d_0+d_1))mq_2} \|\tilde{R}_{l,m}f\|_{p_2}^{q_2} \right)^{\frac{1}{q_2}} &\leq c \|\tilde{u}\|_{B_{p_0,q_0}^{s_0}} \\ &\times \left( \sum_{m=0}^{\infty} 2^{(s-d_1)mq} \|\psi_m(D)\Phi_{m-l}(D)D^{\eta_1}f\|_p^q \right)^{\frac{1}{q}} \\ &\leq c' \|f\|_{B_{p,q}^s} < \infty. \end{aligned} \quad (5.45)$$

Hence  $\sum_{m=0}^{\infty} \tilde{R}_{l,m}f$  converges by Lemma 2.3 (cf (5.41)), so as desired  $\tilde{R}_{l,m}f \rightarrow 0$ . If  $p$  and/or  $p_0$  is in  $]0,1[$  one can use Sobolev embeddings into  $B_{1,q}^{s+n-\frac{n}{p}}$  and  $B_{1,q_0}^{s_0+n-\frac{n}{p_0}}$ , since these spaces also fulfil (5.41).

The rest of  $R_{l,m}f$  may be handled with Lemma 2.5, as done in the  $\pi_1$ - and  $\pi_3$ -parts of  $P_u$  (this is also analogous to the proof of Lemma 2.7). This shows that  $\sum_{m=0}^{\infty} R_{l,m}f$  converges in  $\mathcal{S}'$  so that  $\lim_m R_{l,m}f = 0$ , hence  $\lim_m \sum_l' R_{l,m}f = \lim_m R_m f = 0$ . Hence  $P_u$  is of type 1, 1 as claimed.

5°. If  $g$  is as in the theorem,  $x \in \text{sing supp } \ell_{\Omega}g$  implies that  $x \in \text{sing supp } g \cup \mathbb{R}^n \setminus \Omega$ . By 4° and Theorem 5.14,  $\text{sing supp } \ell_{\Omega}g$  is not enlarged by  $P_u$ , so  $r_{\Omega}P_u \ell_{\Omega}g$  is  $C^{\infty}$  in the part of  $\Omega$  where  $g$  is so.  $\square$

*Remark 5.16.* As indicated above, the theory of type 1, 1 operators is still far from complete. To avoid any ambiguity, the exact parilinearisations have been defined here without reference to these operators, and the Parilinearisation Theorem was for the same reason proved directly, before the factorisation through type 1, 1 operators was established.

*Remark 5.17.* One way to attempt a symbolic calculus would be to replace  $\ell_{\Omega}$  by  $e_{\Omega}$ , ie by extension by zero outside of  $\Omega$ . The resulting linearisation  $\tilde{L}_u$  would have the form  $\tilde{L}_u g = r_{\Omega} P e_{\Omega} g$  where  $P$  is in  $\text{OP}(S_{1,1}^{\omega}(\mathbb{R}^n \times \mathbb{R}^n))$  as in Theorem 5.15. For  $\tilde{L}_u$  to have order  $\omega$  in spaces with  $s > 0$ , it is envisaged that the transmission property would be needed for  $P$ . However, transmission *conditions* have been worked out for  $S_{\rho,\delta}^d$  with  $\delta < 1$ , cf [GH91]. For  $\delta = 1$  there is a fundamental difficulty because  $\text{OP}(S_{1,1}^{\omega})$  in general, cf (5.34), is defined on  $H_p^s$  for  $s > \omega > 0$  — whereas the usual induction proof of the continuity of truncated pseudo-differential operators with transmission property effectively requires application to spaces with  $s < 0$  (in the induction step,  $r_{\Omega}P$  is applied to distributions supported by the boundary  $\Gamma \subset \mathbb{R}^n$ ). Also the powers  $(R_D L_u)^N$  should be covered, so the general rules of composition with the operators in the Boutet de Monvel calculus should be established. All in all this is better investigated elsewhere; it could be useful eg in reductions where traces or solution operators of other problems are applied to the parametrix formula.

## 6. THE VON KARMAN EQUATIONS OF NON-LINEAR VIBRATION

The preceding sections apply to von Karman's equations for a thin, buckling plate, initially filling an open domain  $\Omega \subset \mathbb{R}^2$  with  $C^{\infty}$ -boundary  $\Gamma$ . The following

is inspired by [Lio69, Ch. 1.4] and by the treatise of P. G. Ciarlet [Cia97, Ch. 5], that also settles the applicability of the model to physical problems.

In the stationary case the problem is to find two real-valued functions  $u_1$  and  $u_2$  (displacement and stress) defined in  $\Omega$  and fulfilling

$$\Delta^2 u_1 - [u_1, u_2] = f \quad \text{in } \Omega \quad (6.1a)$$

$$\Delta^2 u_2 + [u_1, u_1] = 0 \quad \text{in } \Omega \quad (6.1b)$$

$$\gamma_k u_1 = 0 \quad \text{on } \Gamma \text{ for } k = 0, 1 \quad (6.1c)$$

$$\gamma_k u_2 = \psi_k \quad \text{on } \Gamma \text{ for } k = 0, 1. \quad (6.1d)$$

Hereby  $\Delta^2$  denotes the biharmonic operator, whilst  $[\cdot, \cdot]$  as in Example 5.2 stands for the bilinear operator

$$[v, w] = D_1^2 v D_2^2 w + D_2^2 v D_1^2 w - 2D_{12}^2 v D_{12}^2 w. \quad (6.2)$$

For the real-valued case with  $\psi_0 = \psi_1 = 0$ , it is well known that Brouwer's fixed point theorem implies the existence of solutions with  $u_j \in F_{2,2}^2(\overline{\Omega})$  for given data  $f \in F_{2,2}^{-2}(\overline{\Omega})$ ; cf [Lio69, Thm. 4.3] and (1.5). For  $\psi_k \in F_{2,2}^{2-k-1/2}(\Gamma)$  solutions are established by non-linear minimisation in [Cia97, Thm. 5.8-3]. Concerning the regularity it was eg shown in [Lio69, Thm. 4.4] that if  $f \in L_p(\Omega)$  for some  $p > 1$ , then any of the above solutions of (6.1) fulfils that  $u_1 \in F_{p,2}^4(\overline{\Omega})$  while  $u_2$  belongs to  $F_{q,2}^4(\overline{\Omega})$  for any  $q < \infty$ . It was also noted in [Lio69] that iteration would give more, eg that the problem is hypoelliptic. Corresponding results for non-trivial  $\psi_0$  and  $\psi_1$  may be found in [Cia97, Thm. 5.8-4].

These results are generalised in three ways in the present paper, as a consequence of the general investigations: firstly the assumptions on the data and on the solution  $(u_1, u_2)$  are considerably weaker, including fully inhomogeneous data; secondly the weak solutions are carried over to a wide range of spaces with  $p \neq 2$ . Thirdly the non-linear terms are shown to have no influence on the solution's regularity (within the Besov and Lizorkin–Triebel scales).

In the discussion of (6.1), the coupling of the two non-linear equations is a little inconvenient, since the Exact Paralinearisation Theorem, 5.7, needs a modification to this situation. But this can be done easily when  $u_1$  and  $u_2$  are given in the same space, for in the proof of Theorem 5.7 the mapping properties will then remain the same regardless of whether  $u_1$  or  $u_2$  is inserted in the various  $\pi_j$ -expressions. For brevity, it is left for the reader to substantiate this expansion of the theorem. (More general methods follow in Section 7.)

Because  $[v, w]$  is of type  $(2, 2)$ , the quadratic standard domain in (5.9) is for  $Q_0(u) := [u, u]$  given by  $s > 2 + (\frac{2}{p} - 1)_+$ , and clearly  $(s, p, q) = (2, 2, 2)$  is at the boundary of and therefore outside of  $\mathbb{D}(Q_0)$ ; cf Figure 3. Hence Theorem 5.7 does barely not apply as it stands.

To carry over weak solutions to other spaces, one can use the more refined estimates for the borderlines in Theorem 5.13. In fact the co-domain of type  $B_{p,\infty}$  is embedded into  $E_{p,q}^{s-\omega-\varepsilon}$  for  $\varepsilon > 0$ , so this gives that  $L_{(u_1, u_2)}$  has order  $\omega = 3 + \varepsilon$  when both  $(s_0, p_0, q_0)$  and  $(s, p, q)$  equal  $(2, 2, 2)$ . For other choices of  $(s, p, q)$

the continuity properties of  $L_{(u_1, u_2)}$  are given by Theorem 5.7. In addition,  $L_{(u_1, u_2)}$  linearises the non-linear terms in (6.1), for at  $(2, 2, 2)$  these only contains products of  $L_2$ -functions, whence the conclusions of Lemma 5.4 remain valid (the assumption  $s > \max(d_0, d_1)$  is then not needed in the proof of the lemma). In this way Theorem 3.2 can be used for the von Karman problem, when  $\mathbb{D}(\mathcal{N})$  is taken as  $\mathbb{D}(Q_0) \cup \{(2, 2, 2)\}$  and  $\mathbb{D}(B_u)$  likewise is the union of  $\mathbb{D}(L_{(u_1, u_2)})$  and  $\{(2, 2, 2)\}$ . (Parameter domains were not required to be open in Theorem 3.2.)

One could also envisage other problems in which the weak solutions belong to spaces at the borderline of the quadratic standard domain, so that results like Theorem 5.13 would be the only manageable way to apply Theorem 3.2.

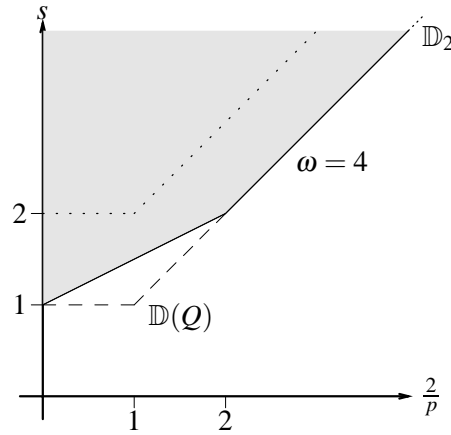


FIGURE 3. The quadratic standard domains of  $Q$  and  $Q_0$  (in dots) in relation to  $\mathbb{D}_2$ .

For the von Karman problem, however, the symmetry properties of  $[v, w]$  make it possible to avoid the rather specialised estimates in Theorem 5.13. Indeed, as recalled in Example 5.2,  $[\cdot, \cdot]$  is a restriction of

$$B(v, w) = D_{12}^2(D_1 v D_2 w + D_2 v D_1 w) - D_1^2(D_2 v D_2 w) - D_2^2(D_1 v D_1 w). \quad (6.3)$$

Since  $B$  is of type  $(1, 1, 2)$ , the larger domain  $\mathbb{D}(Q)$  is given by  $s > 1 + (\frac{2}{p} - 1)_+$  according to (5.9). But by (1.22) the appropriate parameter domain for the linear part is  $\mathbb{D}_2$ , and  $\mathbb{D}(Q) \cap \mathbb{D}_2 = \mathbb{D}_2$ , cf Figure 3.

On the resulting domain  $\mathbb{D}_2$ , the operator  $Q$  is  $\Delta^2$ -moderate in view of Corollary 5.9. It is moreover easy to infer from (5.14) that  $\omega = 4$  holds on the borderline with  $s = 2/p$  (for  $p < 1$ ) of  $\mathbb{D}_2$ .

This leads to the following result on the fully inhomogeneous problem:

**Theorem 6.1.** *Let two functions  $u_1, u_2 \in B_{p,q}^s(\overline{\Omega})$  with  $(s, p, q)$  in  $\mathbb{D}_2$  solve*

$$\Delta^2 u_1 - B(u_1, u_2) = f_1 \quad \text{in } \Omega \quad (6.4a)$$

$$\Delta^2 u_2 + B(u_1, u_1) = f_2 \quad \text{in } \Omega \quad (6.4b)$$

$$\gamma_k u_1 = \varphi_k \quad \text{on } \Gamma \text{ for } k = 0, 1 \quad (6.4c)$$

$$\gamma_k u_2 = \psi_k \quad \text{on } \Gamma \text{ for } k = 0, 1, \quad (6.4d)$$

for data  $f_k \in B_{r,o}^{t-4}(\overline{\Omega})$ , with  $k = 1, 2$ , together with  $\varphi_0, \psi_0 \in B_{r,o}^{t-\frac{1}{r}}(\Gamma)$  and  $\varphi_1, \psi_1 \in B_{r,o}^{t-1-\frac{1}{r}}(\Gamma)$  whereby  $(t, r, o) \in \mathbb{D}_2 \cap \mathbb{D}(L_{(u_1, u_2)})$ , that is

$$\begin{aligned} t &> 1 + \frac{1}{r} + \left(\frac{1}{r} - 1\right)_+, \\ t &> 2 - s + \left(\frac{2}{r} + \frac{2}{p} - 2\right)_+. \end{aligned} \quad (6.5)$$

Then  $u_1, u_2$  belong to  $B_{r,o}^t(\overline{\Omega})$ . If instead  $f_k \in F_{r,o}^{t-4}(\overline{\Omega})$ ,  $\varphi_0, \psi_0 \in B_{r,r}^{t-\frac{1}{r}}(\Gamma)$  and  $\varphi_1, \psi_1 \in B_{r,r}^{t-1-\frac{1}{r}}(\Gamma)$  for some  $(t, r, o)$  fulfilling (6.5), then it follows that  $u_1, u_2 \in F_{r,o}^t(\overline{\Omega})$ .

Since  $\mathbb{D}_2$  is open, it is not a loss of generality here to assume for the Lizorkin–Triebel case that  $u_1$  and  $u_2$  are given in a Besov space.

One can prove the theorem directly, as indicated above, but it will follow from the general considerations in Section 7. So instead the consequences for existence of solutions in Besov and Lizorkin–Triebel spaces are given; this amounts to a solvability theory for the domain bounded by the dotted lines in Figure 3. It is also noteworthy that solutions exist for data with arbitrarily large norms:

**Corollary 6.2.** *Let  $f \in B_{p,q}^{s-4}(\overline{\Omega})$  and  $\psi_k \in B_{p,q}^{s-k-\frac{1}{p}}(\Gamma)$ , for  $k = 0, 1$ , be real-valued data for some  $(s, p, q)$  fulfilling*

$$s > 2 + \left(\frac{2}{p} - 1\right)_+, \quad \text{or} \quad (6.6a)$$

$$s = 2 + \left(\frac{2}{p} - 1\right)_+ \quad \text{and} \quad q \leq 2. \quad (6.6b)$$

Then there exists a solution  $(u_1, u_2)$  in  $B_{p,q}^s(\overline{\Omega})^2$  of the equations in (6.1).

If  $f \in F_{p,q}^{s-4}(\overline{\Omega})$  and  $\psi_k \in B_{p,p}^{s-k-\frac{1}{p}}(\Gamma)$ , for  $k = 0, 1$ , and  $(s, p, q)$  fulfils either (6.6a) or

$$s = 2 + \left(\frac{2}{p} - 1\right)_+, \quad \text{and } q \leq 2 \text{ if } p \geq 2, \quad (6.7)$$

then (6.1) has a solution  $(u_1, u_2)$  in  $F_{p,q}^s(\overline{\Omega})^2$ .

*Proof.* Under the assumptions on  $(s, p, q)$ , the data  $f$  and  $\psi_k$  belong to  $F_{2,2}^{-2}(\overline{\Omega})$  and  $B_{2,2}^{2-k-\frac{1}{2}}(\Gamma)$ , as seen by the usual embeddings. So by invoking [Cia97, Thm. 5.8-3] there is a solution  $(u_1, u_2) \in F_{2,2}^2(\overline{\Omega})^2$ ; according to Theorem 6.1 it also belongs to  $B_{p,q}^s(\overline{\Omega})^2$  or  $F_{p,q}^s(\overline{\Omega})^2$ , respectively.  $\square$

**Example 6.3.** Equation (6.1) may be considered with force term  $f(x_1, x_2)$  equal to  $1(x_1) \otimes \delta_0(x_2)$  and  $0 \in \Omega$ . Such singular data could model displacements and

stresses generated by a heavy rod lying along the  $x_1$ -axis on a table, obtained by clamping a wooden plate along its edges to a sturdy metal frame.

By (2.33), this  $f \in B_{p,\infty}^{\frac{1}{p}-1}(\overline{\Omega})$  for every  $p \in ]0, \infty]$ . So Corollary 6.2 gives for every set of  $\psi_k \in B_{p,\infty}^{3-k}(\Gamma)$ ,  $k = 0, 1$ , with fixed  $p \in ]0, \infty]$ , a solution  $(u_1, u_2)$  in  $B_{p,\infty}^{3+\frac{1}{p}}(\overline{\Omega})^2$  of (6.1). By Theorem 6.1, it belongs to this space for every  $p \in ]0, \infty]$ , when  $\psi_0 = \psi_1 = 0$ .

*Remark 6.4.* Although the coupling of the two non-linear equations in (6.1), as described, could be handled using that  $u_1$  and  $u_2$  are sought after in the same space, it seems more flexible to stick with the general set-up in Section 3 by developing a theory in which the pair  $(u_1, u_2)$  is regarded as the unknown, entering the bilinear form twice. This only requires some projections onto  $u_1$  and  $u_2$ , cf the details around (7.16) below. For this purpose it is convenient to generalise product type operators to a framework of vector bundles, as done in the next section.

## 7. SYSTEMS OF SEMI-LINEAR BOUNDARY PROBLEMS

In this section the abstract results of Section 3 and those on parilinearisation in Section 4 will be carried over to a general framework for semi-linear elliptic boundary problems. This is formulated in a vector bundle set-up, not just because this is natural for linear elliptic systems of multi-order, but also because vector bundles are useful for handling *non-linearities*, as mentioned in Remark 6.4 above.

**7.1. General linear elliptic systems.** Because the parametrix construction relies on a linear theory with the properties in (I)–(II) of Section 3, it is natural to utilise the Boutet de Monvel calculus [BdM71]. The  $L_p$ -results for this are reviewed briefly below (building on [Joh96], that extends  $L_p$ -results of G. Grubb [Gru90] and J. Franke [Fra85, Fra86a]). Introductions to the calculus may be found in [Gru97, Gru91] or [JR97, Sect. 4.1], and a thorough account in [Gru96].

Recall that  $\Omega \subset \mathbb{R}^n$  denotes a smooth, open, bounded set with  $\partial\Omega = \Gamma$ . The main object is then a multiorder Green operator, designated by  $\mathcal{A}$ , ie,

$$\mathcal{A} = \begin{pmatrix} P_\Omega + G & K \\ T & S \end{pmatrix} \quad (7.1)$$

where  $P = (P_{ij})$  and  $G = (G_{ij})$ ,  $K = (K_{ij})$ ,  $T = (T_{ij})$  and  $S = (S_{ij})$ . Here  $i \in I_1 := \{1, 2, \dots, i_\Omega\}$  and  $i \in I_2 := \{i_\Omega + 1, \dots, i_\Gamma\}$ , respectively, in the two rows of the block matrix  $\mathcal{A}$ . Similarly it holds that  $j \in J_1 := \{1, 2, \dots, j_\Omega\}$  and  $j \in J_2 := \{j_\Omega + 1, \dots, j_\Gamma\}$ , respectively, in the two columns of  $\mathcal{A}$ ; that is,  $\mathcal{A}$  is an  $i_\Gamma \times j_\Gamma$  matrix operator with indices belonging to  $I \times J$ , when  $I = I_1 \cup I_2$  and  $J = J_1 \cup J_2$ .

Each  $P_{ij}$ ,  $G_{ij}$ ,  $K_{ij}$ ,  $T_{ij}$  and  $S_{ij}$  belongs to the poly-homogeneous calculus of pseudo-differential boundary problems. More precisely,  $P$  is a pseudo-differential operator satisfying the uniform two-sided transmission condition (at  $\Gamma$ ),  $G$  is a singular Green operator,  $K$  a Poisson and  $T$  a trace operator, while  $S$  is an ordinary pseudo-differential operator on  $\Gamma$ . (The well-known requirements on the symbols and symbol kernels may be found in the references above; they are not recalled, since they will not enter the arguments directly here.) The operator in the  $ij^{\text{th}}$

entry of  $\mathcal{A}$  is taken to be of order  $d + b_i + a_j$ , where  $d \in \mathbb{Z}$ ,  $\mathbf{a} = (a_j) \in \mathbb{Z}^{j_\Gamma}$  and  $\mathbf{b} = (b_i) \in \mathbb{Z}^{i_\Gamma}$ ; for each  $j$ , both  $P_{ij,\Omega} + G_{ij}$  and  $T_{ij}$  is supposed to be of class  $\kappa + a_j$  for some fixed  $\kappa \in \mathbb{Z}$ . For short  $\mathcal{A}$  is then said to be of order  $d$  and class  $\kappa$  (relatively to  $(\mathbf{a}, \mathbf{b})$ , more precisely).

Recall that the transmission condition ensures that  $P_\Omega := r_\Omega P e_\Omega$  has the same order on all spaces on which it is defined. More explicitly this means that each  $P_{ij,\Omega}$  has order  $d + a_j + b_i$  on every  $B_{p,q}^s$  and  $F_{p,q}^s$  with arbitrarily high  $s > \kappa + a_j + 1 - \frac{1}{p}$ ; implying, say that  $C^\infty(\overline{\Omega})$  is mapped into  $C^\infty(\overline{\Omega})$ , without blow-up at  $\Gamma$ . (Thus  $P_\Omega$  has the transmission *property*.)

In general the operators act on spaces of sections of vector bundles  $E_j$  over  $\Omega$  and  $F_j$  over  $\Gamma$ , with  $j$  running in  $J_1$  and  $J_2$ , respectively; they map into sections of other such bundles  $E'_i$  and  $F'_i$ . The fibres of  $E_j, F_j$  have dimension  $M_j, N_j$ , while  $\dim E'_i = M'_i$  and  $\dim F'_i = N'_i$ . Letting

$$V = (E_1 \oplus \cdots \oplus E_{j_\Omega}) \cup (F_{j_\Omega+1} \oplus \cdots \oplus F_{j_\Gamma}) \quad (7.2)$$

$$V' = (E'_1 \oplus \cdots \oplus E'_{i_\Omega}) \cup (F'_{i_\Omega+1} \oplus \cdots \oplus F'_{i_\Gamma}), \quad (7.3)$$

then  $\mathcal{A}$  is a map  $C^\infty(V) \rightarrow C^\infty(V')$ . One may either regard  $C^\infty(V)$  as a short hand for  $C^\infty(E_1) \oplus \cdots \oplus C^\infty(F_{j_\Gamma})$ , or view  $V$  as a vector bundle with the dimension of both the base manifold  $\Omega \cup \Gamma$  and of the fibres over its points  $x$  be depending on whether  $x \in \Omega$  or  $x \in \Gamma$  (as allowed in eg the set-up of [Lan72]). Similarly for  $V'$ .

The following spaces are adapted to the orders and classes of  $\mathcal{A}$ ,

$$B_{p,q}^{s+\mathbf{a}}(V) = \left( \bigoplus_{j \leq j_\Omega} B_{p,q}^{s+a_j}(E_j) \right) \oplus \left( \bigoplus_{j_\Omega < j} B_{p,q}^{s+a_j-\frac{1}{p}}(F_j) \right) \quad (7.4)$$

$$B_{p,q}^{s-\mathbf{b}}(V') = \left( \bigoplus_{i \leq i_\Omega} B_{p,q}^{s-b_i}(E'_i) \right) \oplus \left( \bigoplus_{i_\Omega < i} B_{p,q}^{s-b_i-\frac{1}{p}}(F'_i) \right). \quad (7.5)$$

Here the spaces of  $B_{p,q}^s$ -sections of  $E_j$  etc is defined and normed as usual via local trivialisations.  $F_{p,q}^{s+\mathbf{a}}(V)$  and  $F_{p,q}^{s-\mathbf{b}}(V')$  are analogous ( $p < \infty$ ), except that  $q = p$  in the summands over  $\Gamma$ ; as usual  $F_{p,p}^s(F_j) = B_{p,p}^s(F_j)$  etc. For convenience

$$\|v|B_{p,q}^{s+\mathbf{a}}\| = \left( \|v_1|B_{p,q}^{s+a_1}(E_1)\|^q + \cdots + \|v_{j_\Gamma}|B_{p,q}^{s+a_{j_\Gamma}-\frac{1}{p}}(F_{j_\Gamma})\|^q \right)^{\frac{1}{q}} \quad (7.6)$$

$$\|v|F_{p,q}^{s+\mathbf{a}}\| = \left( \|v_1|F_{p,q}^{s+a_1}(E_1)\|^p + \cdots + \|v_{j_\Gamma}|F_{p,p}^{s+a_{j_\Gamma}-\frac{1}{p}}(F_{j_\Gamma})\|^p \right)^{\frac{1}{p}}, \quad (7.7)$$

with similar conventions for  $B_{p,q}^{s-\mathbf{b}}$  and  $F_{p,q}^{s-\mathbf{b}}$ . With respect to these spaces,  $\mathcal{A}$  is *continuous*

$$\mathcal{A}: B_{p,q}^{s+\mathbf{a}}(V) \rightarrow B_{p,q}^{s-d-\mathbf{b}}(V'), \quad \mathcal{A}: F_{p,q}^{s+\mathbf{a}}(V) \rightarrow F_{p,q}^{s-d-\mathbf{b}}(V'), \quad (7.8)$$

for each  $(s, p, q) \in \mathbb{D}_\kappa$ , when  $p < \infty$  in the Lizorkin–Triebel spaces.

Ellipticity for multi-order Green operators is similar to this notion for single-order operators, except that the principal symbol  $p^0(x, \xi)$  is a matrix with  $p_{ij}^0$  equal to the principal symbol of  $P_{ij}$  *relatively* to the order  $d + b_i + a_j$  of  $P_{ij}$ ; invertibility of  $p^0(x, \xi)$  should hold for all  $x \in \Omega$  and  $|\xi| \geq 1$ . The principal boundary operator  $a^0(x', \xi', D_n)$  is similarly defined and should be invertible as a map from  $\mathcal{S}(\overline{\mathbb{R}}_+)^M \times \mathbb{C}^N$  to  $\mathcal{S}(\overline{\mathbb{R}}_+)^{M'} \times \mathbb{C}^{N'}$  with  $M := \sum_{j \leq j_\Omega} M_j$ ,  $N := \sum_{j_\Omega < j \leq j_\Gamma} N_j$  etc.

For the mapping properties of elliptic systems  $\mathcal{A}$  and their parametrices one has the next theorem, which is an unbridged version of [Joh96, Thm 5.2].

**Theorem 7.1.** *Let  $\mathcal{A}$  denote a multi-order Green operator going from  $V$  to  $V'$ , and of order  $d$  and class  $\kappa$  relatively to  $(\mathbf{a}, \mathbf{b})$  as described above. If  $\mathcal{A}$  is injectively or surjectively elliptic, then  $\mathcal{A}$  has, respectively, a left- or right-parametrix  $\widetilde{\mathcal{A}}$  in the calculus.  $\widetilde{\mathcal{A}}$  can be taken of order  $-d$  and class  $\kappa - d$ , and then  $\widetilde{\mathcal{A}}$  is bounded in the opposite direction in (7.8) for all the parameters  $(s, p, q) \in \mathbb{D}_\kappa$ . The corresponding is true for  $F_{p,q}^{s+\mathbf{a}}(V)$  and  $F_{p,q}^{s-d-\mathbf{b}}(V')$ . In the elliptic case, all these properties hold for  $\mathcal{A}$ , and the parametrices are two-sided.*

The above statement is deliberately rather brief. It should be added that (7.8) is sharp, since it only holds for  $(s, p, q)$  outside  $\overline{\mathbb{D}}_\kappa$  if the class is effectively lower than  $\kappa$ . Moreover, the kernel of  $\mathcal{A}$  is a finite-dimensional space in  $C^\infty(V)$ , which is the same for all  $(s, p, q)$  and in the  $B$ - and  $F$ -cases; the range is closed with complements that can be chosen to have similar properties. The reader is referred to [Gru90, Joh96] for this. In particular the  $(s, p, q)$ -invariance of the range complements implies that the compatibility conditions on the data are fulfilled for all  $(s, p, q)$ , if they are so for one parameter. Hence these conditions can be ignored in the following regularity investigations.

For the inverse regularity properties of an injectively elliptic system  $\mathcal{A}$ , note that, by the above theorem, the left-parametrix  $\widetilde{\mathcal{A}}$  may be chosen so that  $\mathcal{R} := I - \widetilde{\mathcal{A}}\mathcal{A}$  has class  $\kappa$  and order  $-\infty$ , hence is continuous

$$\mathcal{R}: B_{p,q}^{s+\mathbf{a}}(V) \rightarrow C^\infty(V) \quad \text{for every } (s, p, q) \in \mathbb{D}_\kappa. \quad (7.9)$$

So if  $\mathcal{A}u = f$  for some  $u \in B_{p_1,q_1}^{s_1+\mathbf{a}}(V)$  and data  $f \in B_{p_0,q_0}^{s_0-d-\mathbf{b}}(V')$ , and if  $(s_j, p_j, q_j)$  belongs to  $\mathbb{D}_\kappa$  for  $j = 0$  and  $1$ , then application of  $\mathcal{A}$  to  $\mathcal{A}u = f$  yields (cf (1.8)–(1.10) ff)

$$u = \widetilde{\mathcal{A}}f + \mathcal{R}u \in B_{p_0,q_0}^{s_0+\mathbf{a}}(V). \quad (7.10)$$

It can now be explicated how this framework fits with the conditions (I)–(II) of Section 3: for each fixed  $q \in ]0, \infty]$  let  $\mathbb{S} = \{(s, p) \mid s \in \mathbb{R}, 0 < p \leq \infty\}$  and take

$$X_p^s = B_{p,q}^{s+\mathbf{a}}(V), \quad Y_p^s = B_{p,q}^{s-\mathbf{b}}(V'), \quad A_{(s,p)} = \mathcal{A}|_{B_{p,q}^{s+\mathbf{a}}(V)}, \quad \mathbb{D}(A) = \mathbb{D}_\kappa. \quad (7.11)$$

Moreover,  $\widetilde{A} = \widetilde{\mathcal{A}}$  should be chosen to be of class  $\kappa - d$ . For the corresponding spaces  $X_p^s = F_{p,q}^{s+\mathbf{a}}(V)$  and  $Y_p^s = F_{p,q}^{s-\mathbf{b}}(V')$  one needs a little precaution because the sum and integral exponents in (7.7) are equal in the spaces over the boundary bundles  $F_j$ . Then (3.3) is not a direct consequence of (2.8) ff, but for  $p > r$ ,

$$F_{p,p}^{s+a_j-\frac{1}{p}}(F_j) \hookrightarrow F_{r,p}^{s+a_j-\frac{1}{p}}(F_j) \hookrightarrow F_{r,r}^{s+a_j-\frac{1}{r}}(F_j). \quad (7.12)$$

In this way (I) and (II) holds also for these spaces.

**Example 7.2.** For the Dirichlét problem for  $\Delta^2$ , which enters the von Karman equations, it is natural to let

$$\mathcal{A} = \begin{pmatrix} \Delta^2 & 0 \\ 0 & \Delta^2 \\ \gamma_0 & 0 \\ \gamma_1 & 0 \\ 0 & \gamma_0 \\ 0 & \gamma_1 \end{pmatrix}, \quad (7.13)$$

whereby  $d = 4$ ,  $\kappa = 2$ ,  $\mathbf{a} = (0, 0)$  and  $\mathbf{b} = (0, 0, -4, -3, -4, -3)$ . The choice in (7.11) amounts to

$$X_p^s = B_{p,q}^s(\overline{\Omega})^2 \quad (7.14)$$

$$Y_p^{s-4} = B_{p,q}^{s-4}(\overline{\Omega})^2 \oplus (B_{p,q}^{s-\frac{1}{p}}(\Gamma) \oplus B_{p,q}^{s-1-\frac{1}{p}}(\Gamma))^2; \quad (7.15)$$

this is clear since one can use the trivial bundles  $V = \Omega \times \mathbb{C}^2$  and  $V' = (\Omega \times \mathbb{C}^2) \cup (\Gamma \times \mathbb{C})^4$  for this problem.

**7.2. General product type operators.** Together with the Green operator  $\mathcal{A}$  in (7.13) above, a treatment of the von Karman equation may conveniently use the bilinear operator  $\tilde{B}$  given on  $v = (v_1, v_2)$  and  $w = (w_1, w_2)$  by

$$\tilde{B}(v, w) = \begin{pmatrix} -[v_1, w_2] & [v_1, w_1] & 0 & 0 & 0 & 0 \end{pmatrix}^T. \quad (7.16)$$

Indeed, in the set-up of the previous section, a solution  $u = (u_1, u_2)$  of (6.1) is a section of the trivial bundle  $\Omega \times \mathbb{C}^2$ , of which the two canonical projections  $u_1$  and  $u_2$  enter directly into the expressions in (6.1). The same projections enter for  $v = w = u$  in (7.16) above, and this is taken as the guiding principle in a generalisation of product type operators to vector bundles.

Between vector bundles, a product type operator is roughly just an operator that locally has the form introduced in Section 5. But in relation to a given elliptic system  $\mathcal{A}$  of order  $d$  and class  $\kappa$  with respect to a fixed set of integers  $(\mathbf{a}, \mathbf{b})$ , it is useful to introduce a class of product type operators with compatible mapping properties.

Since the non-linearities typically send sections over  $\Omega$  to other such sections (so that sections over  $\Gamma$  and zero-entries as in (7.16) can be tacitly omitted), the following framework should suffice for most applications:

Given bundles over  $\Omega$  as in (7.2)–(7.3), there are bundles

$$W = E_1 \oplus \cdots \oplus E_{j_\Omega}, \quad W' = E'_1 \oplus \cdots \oplus E'_{i'_\Omega}, \quad (7.17)$$

$$\beta_j: E_j \rightarrow \Omega, \quad \beta'_i: E'_i \rightarrow \Omega \quad (7.18)$$

in which sections  $w$  and  $w'$ , respectively, may naturally be regarded as  $j_\Omega$ - and  $i'_\Omega$ -tuples of sections (by means of projections  $\text{pr}_j$  and  $\text{pr}'_i$ )

$$w = (w_1, \dots, w_{j_\Omega}), \quad w' = (w'_1, \dots, w'_{i'_\Omega}). \quad (7.19)$$



There is also a finite covering  $\Omega = \bigcup U_\kappa$  of local coordinate systems  $\kappa: U_\kappa \rightarrow \tilde{U}_\kappa$ , for disjoint open balls or half balls  $\tilde{U}_\kappa$  in  $\mathbb{R}^n$ . Alternatively  $\tilde{U}_\kappa$  is written  $U_{\tilde{\kappa}}$ , as it is the domain of  $\tilde{\kappa} := \kappa^{-1}$ ; then  $E_{p,q}^s(\overline{U_{\tilde{\kappa}}})$  denotes the function spaces over  $\tilde{U}_\kappa$ .

With this there are associated trivialisations  $\tau_{j\kappa}$  and  $\tau'_{i\kappa}$ , for each  $j, i$  and  $\kappa$ , together with associated projections  $\text{pr}_{j\kappa m}$  onto the  $m^{\text{th}}$  coordinate of  $\mathbb{C}^{M_j}$ :

$$\beta_j^{-1}(U_\kappa) \xrightarrow{\tau_{j\kappa}} \tilde{U}_\kappa \times \mathbb{C}^{M_j} \xrightarrow{\text{pr}_{j\kappa m}} \mathbb{C}. \quad (7.20)$$

For short,  $\tau_{j\kappa m} := \text{pr}_{j\kappa m} \circ \tau_{j\kappa} \circ \text{pr}_j$ , and similarly for  $\tau'_{i\kappa m}$  and  $\text{pr}'_{i\kappa m}$  in the sequel.

**Definition 7.3.** An operator  $B$  from  $W \oplus W$  to  $W'$  is of product type  $(d_0, d_1, d_2)$  compatibly with integers  $(\mathbf{a}, \mathbf{b})$  as in (7.4)–(7.5) ff if the following holds:

- (i) Each map  $\tau'_{i\kappa m} B(v, w)$  can be written

$$\tau'_{i\kappa m} B(u, v) = \sum_{j_0, m_0, j_1, m_1} B_{i\kappa m}^{j_0 \kappa m_0, j_1 \kappa m_1}(\tau_{j_0 \kappa m_0}(u), \tau_{j_1 \kappa m_1}(v)), \quad (7.21)$$

where  $B_{i\kappa m}^{j_0 \kappa m_0, j_1 \kappa m_1}$  maps pairs of sections of  $W$  to sections of  $\tilde{U}_\kappa \times \mathbb{C}$  and only depends on two projections  $\tau_{j_0 \kappa m_0}(v)$  and  $\tau_{j_1 \kappa m_1}(w)$ , where  $1 \leq m_0 \leq M_{j_0}$  and  $1 \leq m_1 \leq M_{j_1}$ .

- (ii) Each  $B_{i\kappa m}^{j_0 \kappa m_0, j_1 \kappa m_1}$  is of product type  $(d_0 + a_{j_0}, d_1 + a_{j_1}, d_2 + b_i)$  on the open set  $\tilde{U}_\kappa$  of  $\mathbb{R}^n$ .

*Remark 7.4.* The non-linear operator  $\tilde{B}$  in (7.16), that enters the von Karman equation, has the structure in Definition 7.3. Indeed, working in  $\Omega \times \mathbb{C}^2$  one has  $i = j = 1$ , but the choice  $m = 1$  in (i) gives  $-[v_1, w_2]$  (if  $\Omega$  is flat such as a ball), so that the non-trivial terms in (7.21) have  $m_0 = 1, m_1 = 2$ ; whilst  $m = 2$  gives  $m_0 = m_1 = 1 \neq m$ .

As another illustration, the finite sums appear directly in the Navier–Stokes equation, where the unknown  $(u, p)$  is a section of  $W = W' = (\Omega \times \mathbb{C}^n) \oplus (\Omega \times \mathbb{C})$ , at least for the Dirichlet condition. Here  $(u, p)$  enters the non-linear term  $((u \cdot \nabla)u, 0)$ . For  $i = 1$  each  $m$  gives rise to the sum  $\sum_{m_0=1}^n v_{m_0} \partial_{m_0} w_m$ , where obviously any  $m_0 \in \{1, \dots, n\}$  occurs and  $m_1 = m$ . (For  $i = 2$  the zero-operator appears.)

In the next result pseudo-local operators are defined as usual to be those that decrease or preserve singular supports; the singular support of eg a section  $v$  of  $W$  is the complement in  $\Omega$  of the  $x$  for which  $\tau_{j\kappa m} \circ v$  is  $C^\infty$  from a neighbourhood of  $x$  to  $\mathbb{C}$ , for all  $U_\kappa \ni x$  and all  $j$  and  $m$ . It is understood that universal extension operators have been chosen for the sets  $\tilde{U}_\kappa$ , so the exact parilinearisations are meaningful on these sets.

**Theorem 7.5.** Let  $B$  be of product type  $(d_0, d_1, d_2)$  compatibly with  $(\mathbf{a}, \mathbf{b})$  and with  $d_0 \leq d_1$ ; and let Besov and Lizorkin–Triebel spaces be defined as in (7.4)–(7.5) ff; with the unified notation  $E_{p,q}^{s+\mathbf{a}}(V)$  and  $E_{p,q}^{s-\mathbf{b}}(V')$ . Then  $Q(v) := B(v, v)$  is bounded

$$E_{p,q}^{s+\mathbf{a}}(V) \rightarrow E_{p,q}^{s-\sigma(s,p,q)-\mathbf{b}}(V') \quad \text{for every } (s, p, q) \in \mathbb{D}(Q), \quad (7.22)$$

whereby  $\mathbb{D}(Q)$  and  $\sigma(s, p, q)$  are given by (5.9) and (5.11), respectively.

Moreover, for each  $u \in E_{p_0, q_0}^{s_0 + \mathbf{a}}(V)$  there is a moderate linearisation  $L_u$ , which with  $\omega$  as in (5.14) is bounded

$$L_u : E_{p, q}^{s + \mathbf{a}}(V) \rightarrow E_{p, q}^{s - \omega - \mathbf{b}}(V') \quad (7.23)$$

for every  $(s, p, q)$  in the parameter domain  $\mathbb{D}(L_u)$  given by (5.12). Furthermore,  $L_u$  is pseudo-local on every such  $E_{p, q}^{s + \mathbf{a}}(V)$ .

*Proof.* Let  $(s_0, p_0, q_0)$  and  $u \in E_{p_0, q_0}^{s_0 + \mathbf{a}}(V)$  be given; and consider  $(s, p, q)$  such that (5.12) holds. For each pair of projections  $\tau_{j_0 \kappa m_0}(u) \in E_{p_0, q_0}^{s_0 + a_{j_0}}(\overline{U_{\tilde{\kappa}}})$  and  $\tau_{j_1 \kappa m_1}(v)$ , Theorem 5.7 applies to spaces with parameters  $(s_0 + a_{j_0}, p_0, q_0)$  and  $(s + a_{j_1}, p, q)$  since by (ii) the orders are  $d_0 + a_{j_0}$  and  $d_1 + a_{j_1}$ , so there is a  $u$ -dependent linear operator  $L_{i \kappa m}^{j_0 \kappa m_0, j_1 \kappa m_1}$  sending  $E_{p, q}^{s + a_{j_1}}(\overline{U_{\tilde{\kappa}}})$  continuously to  $E_{p, q}^{s - \tilde{\omega}}(\overline{U_{\tilde{\kappa}}})$  for

$$\tilde{\omega} = (d_2 + b_i) + (d_1 + a_{j_1}) + \left(\frac{n}{p_0} - s_0 + d_0\right)_+ + \varepsilon \quad (\varepsilon \geq 0). \quad (7.24)$$

Therefore  $L_{i \kappa m}^{j_0 \kappa m_0, j_1 \kappa m_1}$  is bounded  $E_{p, q}^{s + a_{j_1}}(\overline{U_{\tilde{\kappa}}}) \rightarrow E_{p, q}^{s - \omega - b_i}(\overline{U_{\tilde{\kappa}}})$  for  $\omega$  as in (5.14). In case  $(s_0, p_0, q_0)$  is in the domain  $\mathbb{D}(Q)$ , one can take  $(s, p, q) = (s_0, p_0, q_0)$  without violating (5.12), and then

$$L_{i \kappa m}^{j_0 \kappa m_0, j_1 \kappa m_1}(\tau'_{j_1 \kappa m_1}(u)) = B_{i \kappa m}^{j_0 \kappa m_0, j_1 \kappa m_1}(\tau_{j_0 \kappa m_0}(u), \tau_{j_1 \kappa m_1}(u)). \quad (7.25)$$

Summation over all  $j_0, m_0$  and  $j_1, m_1$  as in (7.21) gives

$$\tau'_{i \kappa m} B(u, v) = \sum L_{i \kappa m}^{j_0 \kappa m_0, j_1 \kappa m_1}(\tau'_{j_1 \kappa m_1}(v)). \quad (7.26)$$

This determines a linear operator  $L_{i \kappa, u}$ , which in the set of sections of  $U_{\tilde{\kappa}} \times \mathbb{C}^{M'_i}$  is given by

$$L_{i \kappa, u}(v) = \left( \sum L_{i \kappa m}^{j_0 \kappa m_0, j_1 \kappa m_1}(\tau'_{j_1 \kappa m_1}(v)) \right)_{m=1, \dots, M'_i}. \quad (7.27)$$

As a composite map,  $L_{i \kappa, u}(v)$  is continuous  $E_{p, q}^{s + \mathbf{a}}(V) \rightarrow E_{p, q}^{s - \omega - b_i}(U_{\tilde{\kappa}})^{M'_i}$ .

Using a partition of unity  $1 = \sum_{\kappa} \psi_{\kappa}$  subordinate to the coordinate patches  $U_{\tilde{\kappa}}$ , there is a bounded linear operator  $L_u : E_{p, q}^{s + \mathbf{a}}(V) \rightarrow E_{p, q}^{s - \omega - \mathbf{b}}(V')$  given by

$$L_u(v)_i = \sum_{\kappa} (\tau'_{i \kappa})^{-1} \circ L_{i \kappa, u}(\psi_{\kappa} v), \quad \text{for } i \in I_1. \quad (7.28)$$

It follows from Theorem 5.15 that each  $L_{i \kappa, u}$  is pseudo-local; and so is  $L_u$ , since the class of pseudo-local operators is closed under addition.

When  $(s_0, p_0, q_0)$  belongs to the domain  $\mathbb{D}(Q)$  given by (5.9), then  $v = u$  is possible for  $(s_0, p_0, q_0) = (s, p, q)$ , and using (7.28)–(7.25),

$$L_u(u)_i = \sum_{\kappa} (\tau'_{i \kappa})^{-1} \circ \tau'_{i \kappa} B(u, \psi_{\kappa} u) = \text{pr}_i B(u, \sum_{\kappa} \psi_{\kappa} u) = \text{pr}_i B(u, u). \quad (7.29)$$

Moreover, the value of  $\omega$  equals  $\sigma(s, p, q)$ , so (7.22) is also proved.  $\square$

**7.3. Semi-linear elliptic systems.** It is now straightforward to specialise Theorem 3.2 to the vector bundle framework of multi-order systems.

For generality's sake it is observed that it suffices, by (II), to take the linear part  $\mathcal{A}$  injectively elliptic, ie with a left parametrix  $\widetilde{\mathcal{A}}$  and regularising operator  $\mathcal{R} := I - \widetilde{\mathcal{A}}\mathcal{A}$ . Recall that for a product type operator  $B$ , the linearisation  $L_u$  of  $Q(u) := B(u, u)$  furnished by Theorem 7.5 enters the parametrix

$$P^{(N)} = I + \widetilde{\mathcal{A}}L_u + \cdots + (\widetilde{\mathcal{A}}L_u)^{N-1}. \quad (7.30)$$

As above,  $\mathbb{D}(\mathcal{A}, Q) = \{(s, p, q) \in \mathbb{D}_\kappa \cap \mathbb{D}(Q) \mid \sigma(s, p, q) < d\}$  is the domain where  $Q$  is  $\mathcal{A}$ -moderate. Using these ingredients, one has the following main result:

**Theorem 7.6.** *Let  $\mathcal{A}$  be an injectively elliptic Green operator of order  $d$  and class  $\kappa$  relatively to  $(\mathbf{a}, \mathbf{b})$ , and assume that  $B$  is of product type  $(d_0, d_1, d_2)$  compatibly with  $(\mathbf{a}, \mathbf{b})$ , and with  $d_0 \leq d_1$ , so that  $Q$  has order function  $\sigma(s, p, q)$  on  $\mathbb{D}(Q)$  and moderate linearisations  $L_u$ , according to Theorem 7.5.*

*For a section  $u$  of  $B_{p_0, q_0}^{s_0+\mathbf{a}}(V)$  with  $(s_0, p_0, q_0) \in \mathbb{D}(\mathcal{A}, Q)$ , and any choice  $\widetilde{\mathcal{A}}$  of a left parametrix of  $\mathcal{A}$  of class  $\kappa - d$ , the parametrices  $P^{(N)}$  in (7.30) are bounded endomorphisms on  $B_{p, q}^{s+\mathbf{a}}(V)$  for every  $(s, p, q)$  in  $\mathbb{D}_\kappa \cap \mathbb{D}(L_u)$ . And for  $(s_1, p_1, q_1)$  and  $(s_2, p_2, q_2)$  in  $\mathbb{D}_\kappa \cap \mathbb{D}(L_u)$  the linear operator  $(\widetilde{\mathcal{A}}L_u)^N$  maps  $B_{p_1, q_1}^{s_1+\mathbf{a}}(V)$  to  $B_{p_2, q_2}^{s_2+\mathbf{a}}(V)$  for all sufficiently large  $N$ . If such a section  $u$  solves the equation*

$$\mathcal{A}u + Q(u) = f \quad (7.31)$$

*for data  $f \in B_{r, o}^{t-d-\mathbf{b}}(V')$  with  $(t, r, o) \in \mathbb{D}_\kappa \cap \mathbb{D}(L_u)$ , then*

$$u = P^{(N)}(\widetilde{\mathcal{A}}f + \mathcal{R}u) + (\widetilde{\mathcal{A}}L_u)^N u \quad (7.32)$$

*and  $u \in B_{r, o}^{t+\mathbf{a}}(V)$ . Analogous results are valid for the scales  $F_{p, q}^{s+\mathbf{a}}(V)$  and  $F_{p, q}^{s-\mathbf{b}}(V')$ .*

*Proof.* As observed in (7.11), the choice  $X_p^s = B_{p, q}^{s+\mathbf{a}}(V)$  and  $Y_p^s = B_{p, q}^{s-\mathbf{b}}(V')$  makes conditions (I) and (II) satisfied. As the  $B_u$  in (III) one can take  $L_u$ , for its construction via paramultiplication implies that it is unambiguously defined on intersections of the form  $X_p^s \cap X_{p'}^{s'}$ . Similarly there is commutative diagrams for  $\mathcal{A}$  and  $\widetilde{\mathcal{A}}$  by the general constructions in the Boutet de Monvel calculus and the results in Section 4.3.

Moreover,  $\mathbb{D}(\mathcal{A}, Q)$  is connected and  $\delta = d - \omega(s, p, q)$  is constant and positive; hence (IV) and (V) hold. The claims on  $P^{(N)}$  may now be read off from Theorem 3.2. For  $(\widetilde{\mathcal{A}}L_u)^N$  the sum exponents should also be controlled, but  $\mathbb{D}_\kappa \cap \mathbb{D}(L_u)$  is open, hence contains  $(s_1 - \varepsilon, p_1, q_2)$  for  $\varepsilon > 0$ , so that the larger space  $B_{p_1, q_2}^{s_1-\varepsilon+\mathbf{a}}(V)$  is mapped into  $B_{p_2, q_2}^{s_2+\mathbf{a}}(V)$  for all sufficiently large  $N$ , according to Theorem 3.2.

Finally, since  $(s_0 - \varepsilon, p_0, q_0)$  also belongs to  $\mathbb{D}_\kappa \cap \mathbb{D}(L_u)$  for sufficiently small  $\varepsilon > 0$ , one can assume  $q_0 = o$ . So according to Theorem 3.2 the section  $u$  fulfils (7.32) and belongs to  $X_r^t = B_{r, o}^{t+\mathbf{a}}(V)$ .  $\square$

It should be mentioned that while the abstract framework in Theorem 3.2 was formulated with only  $s$  and  $p$  as parameters, for convenience, the third parameter  $q$  was easily handled in the proof above by simple embeddings.

From the given examples it is clear that Theorem 6.1 on the von Karman problem is just a special case of the above result. One also has

**Corollary 7.7.** *For operators  $\mathcal{A}$  and  $B$  as in Theorem 7.6, the equation*

$$\mathcal{A}u + Q(u) = f \quad (7.33)$$

*is hypoelliptic, ie for  $f$  in  $C^\infty(V')$  any solution  $u$  belongs to  $C^\infty(V)$ .*

As an application of the parametrix formula (7.32) it is shown that this corollary has a sharper local version. This also uses the obvious fact that the class of pseudo-local maps is stable under composition, in particular  $\widetilde{\mathcal{A}}L_u$  is pseudo-local. (This really only involves the  $P_\Omega + G$ -part of  $\widetilde{\mathcal{A}}$ , since  $L_u$  goes from  $W$  to  $W'$ . And the pseudo-differential part clearly inherits pseudo-locality from the operators on  $\mathbb{R}^n$ , since  $P_\Omega = r_\Omega Pe_\Omega$ . For the singular Green part one can extend [Gru96, Cor. 2.4.7] by means of Rem. 2.4.9 there on  $(x_n, y_n)$ -dependent singular Green operators to get the pseudo-local property. Details are omitted since it is outside of the subject.)

Let  $\Xi \subset \Omega$  be an open subregion with positive distance to the boundary, that is  $\Xi \Subset \Omega$ . Then, if  $f$  in (7.31) in addition fulfils  $f \in B_{r_1, o_1}^{t_1-d-b}(V'_\Xi; \text{loc})$  it will be shown for any solution  $u$  of (7.31) that  $u \in B_{r_1, o_1}^{t_1+a}(V_\Xi; \text{loc})$ .

More precisely,  $f \in B_{r_1, o_1}^{t_1-d-b}(V'_\Xi; \text{loc})$  means that  $\phi f$  is in  $B_{r_1, o_1}^{t_1-d-b}(V')$  for every  $\phi$  in  $C^\infty(\overline{\Omega})$  with compact support contained in  $\Xi$ . Hereby  $\phi f$  is calculated fibrewisely for the components of  $f$ , both in the bundles  $E'_i$  over  $\Omega$ , for  $i \leq i_\Omega$ , and in the  $F'_i$  over  $\Gamma$ , for  $i_\Omega < i \leq i_\Gamma$  (the last part is always 0 for  $\Xi \Subset \Omega$ ). That  $u \in B_{r_1, o_1}^{t_1+a}(V_\Xi; \text{loc})$  is defined similarly, and these conventions extend to the  $F$ -spaces.

**Theorem 7.8.** *Under hypotheses as in Theorem 7.6, suppose  $f \in E_{r_1, o_1}^{t_1-d-b}(V'_\Xi; \text{loc})$  holds in addition to (7.31) for some  $(t_1, r_1, o_1)$  in  $\mathbb{D}_K \cap \mathbb{D}(L_u)$ , for an open set  $\Xi \Subset \Omega$ . Then  $u$  is also a section of  $E_{r_1, o_1}^{t_1+a}(V_\Xi; \text{loc})$ .*

*Proof.* Let  $\psi, \chi_0$  and  $\chi_1 \in C^\infty(\overline{\Omega})$  be chosen so that  $\text{supp } \chi_1 \subset \Xi$  and

$$\chi_0 + \chi_1 \equiv 1, \quad \chi_j \equiv j \text{ on a neighbourhood of } \text{supp } \psi. \quad (7.34)$$

By the parametrix formula (7.32),

$$\psi u = \psi P^{(N)}(\widetilde{\mathcal{A}}(\chi_1 f) + \mathcal{R}u) + \psi P^{(N)}\widetilde{\mathcal{A}}(\chi_0 f) + \psi(\widetilde{\mathcal{A}}L_u)^N u \quad (7.35)$$

and here the last term belongs to  $E_{r_1, o_1}^{t_1+a}(V)$  for a sufficiently large  $N$ , according to the first part of Theorem 7.6. Since  $\widetilde{\mathcal{A}}L_u$  is pseudo-local so is  $P^{(N)}$ , and therefore the inclusion  $\text{sing supp } \widetilde{\mathcal{A}}(\chi_0 f) \subset \text{supp } \chi_0$  implies that  $\psi P^{(N)}\widetilde{\mathcal{A}}(\chi_0 f)$  is in  $C_0^\infty(V) \subset E_{r_1, o_1}^{t_1+a}(V)$ . And because  $\widetilde{\mathcal{A}}(\chi_1 f) + \mathcal{R}u$  is in  $E_{r_1, o_1}^{t_1+a}(V)$ , the fact that  $P^{(N)}$  has order zero gives that also the first term on the right hand side of (7.35) is in  $E_{r_1, o_1}^{t_1+a}(V)$ .  $\square$

When  $\Xi$  adheres to the boundary of  $\Omega$  one can depart from the parametrix formula in the same way. But it seems to require more techniques to show that  $\psi P^{(N)}\widetilde{\mathcal{A}}(\chi_0 f)$  is in  $E_{r_1, o_1}^{t_1+a}(V)$ , for although this term is in  $C^\infty(V)$ , a possible blow-up at the boundary should be ruled out.

## 8. FINAL REMARKS

To sum up, a semi-linear elliptic boundary problem of product type as in (7.31) can conveniently be treated by determining

- $\mathbb{D}(\mathcal{A})$ : to have well-defined boundary conditions, ie  $\mathbb{D}(\mathcal{A}) = \mathbb{D}_\kappa$  when  $\mathcal{A}$  is of class  $\kappa$ , cf (1.22);
- $\mathbb{D}(Q)$ : the quadratic standard domain of  $Q$ , cf (5.9);
- $\mathbb{D}(\mathcal{A}, Q)$ : the domain where  $Q$  is  $\mathcal{A}$ -moderate, obtained from  $\mathbb{D}(\mathcal{A}) \cap \mathbb{D}(Q)$  and the inequality  $s > \frac{n}{p} - d + d_0 + d_1 + d_2$  (unnecessary if  $d_1 - d_0 \geq n$ ), cf (5.23);
- $\mathbb{D}(L_u)$ : the domain of the exact parilinearisation at  $u$ , that is given by the inequality  $s > -s_0 + d_0 + d_1 + (\frac{n}{p} + \frac{n}{p_0} - n)_+$ , cf (5.12);
- $\mathbb{D}_u$ : equal to  $\mathbb{D}(\mathcal{A}) \cap \mathbb{D}(L_u)$ , ie the domain where the parametrics  $P_u^{(N)}$  induced by a given solution  $u$  are defined and the parametrix formula (7.32) holds.

Stated briefly, any given solution  $u$  in  $\mathbb{D}(\mathcal{A}, Q)$  then leads to the parametrix formula (7.32), and  $u$  belongs to any space associated with the data, as long as this space is in the larger domain  $\mathbb{D}_u$ . Theorems 7.6–7.8 contain the precise statements, including hypoellipticity and local properties in subregions  $\Xi \Subset \Omega$ .

**8.1. A last example.** The use of parameter domains is finally illustrated by the following polyharmonic Dirichlét problem perturbed by  $Q(u) = u^2$ , and with  $\gamma u = (\gamma_1 u, \dots, \gamma_{m-1} u)$ :

$$(-\Delta)^m u + u^2 = cf \quad \text{in } \Omega \subset \mathbb{R}^n \quad (8.1a)$$

$$\gamma u = 0 \quad \text{on } \Gamma. \quad (8.1b)$$

Data are taken as a constant  $c > 0$  times the function  $f(x) = |x_1^2 + \dots + x_k^2|^{a/2}$  in a domain  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , with  $\Omega \ni 0$ . For  $a \in ]-k, 0[$  it is clear that  $|x'|^a$  is locally integrable on  $\mathbb{R}^k$ , hence is in  $\mathcal{D}'(\mathbb{R}^k)$ , so by Proposition 2.10  $f$  is in  $B_{p,\infty}^{k/p+a}(\overline{\Omega})$  for all  $p > 0$ .

Now  $(-\Delta)^m : H_0^m(\overline{\Omega}) \rightarrow H^{-m}(\overline{\Omega})$  is a bijection by Lax–Milgram’s lemma, and  $f \in B_{2,\infty}^{k/2+a} \subset H^{-m}$  for  $k + 2a + 2m > 0$ , so under this condition data are consistent with the linear problem; because  $H^m = B_{2,2}^m$  this means that  $(m, 2, 2) \in \mathbb{D}(\Delta_\gamma^m) = \mathbb{D}_m$ . (Here  $\Delta_\gamma^m$  denotes the realisation of  $(-\Delta)^m$  induced by the condition  $\gamma u = 0$ .)

If moreover  $Q$  is of order  $< 2m$  on  $H_0^m$ , ie  $(m, 2, 2)$  is in  $\mathbb{D}(\Delta_\gamma^m, Q)$ , that by (5.23) holds for  $m > n/6$ , then (8.1) is by Proposition 3.3 solvable for certain  $c > 0$ .

However, it is a consequence of the theory here that any solution  $u$  in  $H_0^m$  also is an element of  $W_1^{2m}(\overline{\Omega})$ . This is an improvement in the sense that any derivative  $D^\alpha u$  with  $|\alpha| \leq 2m$  is a function, which is not true for every element of  $H^m$ .

**Theorem 8.1.** *Let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ) be smooth open and bounded,  $0 \in \Omega$ . When  $m \geq n/6$ ,  $k \in \{1, \dots, n\}$  and  $-k < a < 0$ , then problem (8.1) with  $f(x) = c|x'|^a$  has a solution  $u \in H_0^m(\overline{\Omega})$  for sufficiently small  $c > 0$ . Every solution in  $H^m(\overline{\Omega})$  is then also in  $B_{1,\infty}^{k+a+2m}(\overline{\Omega})$ , which is a subspace of  $W_1^{2m}(\overline{\Omega})$ .*

*Proof.* Solvability was noted above. To see that any solution  $u$  in  $H^m \subset B_{2,\infty}^m$  is in  $B_{1,\infty}^{k+a+2m}$ , note first that  $f \in B_{1,\infty}^{k+a}$  by the above. Moreover, to see that the parameter  $(k+a+2m, 1, \infty)$  is in  $\mathbb{D}(L_u)$ , it suffices to apply (5.12) with  $(s_0, p_0, q_0) = (m, 2, \infty)$ , that yields  $k+a+3m > n/2$ . This inequality is fulfilled since  $m > n/6$  and  $k+a > 0$  are assumed. So by Theorem 7.6,  $u$  is in  $B_{1,\infty}^{k+a+2m} \subset W_1^{2m}$ .  $\square$

In dimensions  $n \in \{2, 3, 4, 5\}$  the theorem allows  $m = 1$ , hence covers eg the Dirichlét problem of  $-\Delta$  for any choice of  $k$ , and every  $a \in ]-k, 0[$ . For  $n \in \{6, \dots, 11\}$  the requirement that  $m > n/6$  shows that one gets the  $W_1^{2m}$  regularity at least for  $m = 2$ , ie for the biharmonic Dirichlét problem; etc in higher dimensions.

*Remark 8.2.* In many cases the square  $Q(u) = u^2$  is ill-defined on the ‘target’ space  $B_{1,\infty}^{k+a+2m}$ , for this space is outside of  $\mathbb{D}(Q)$  if (5.9) is violated, ie if  $k+a+2m \leq n/2$ . But by taking  $k+a > 0$  close to 0, it will be enough to have  $m < n/4$ , so there are examples of such target spaces whenever  $m$  can be taken in  $] \frac{n}{6}, \frac{n}{4} [ \cap \mathbb{N}$ , which is non-empty for  $n \in \{5, 9, 10, 11\}$  and for  $n \geq 13$ . For the slightly larger space  $W_1^{2m}$  one can refer to Remark 4.4 for a specific proof that  $Q$  cannot be continuous from  $W_1^{2m} \rightarrow \mathcal{D}'$  for  $m < n/4$ . Note that the result is sharp: if it could be shown that  $u \in B_{1,\infty}^t(\overline{\Omega})$  for  $t$  so large that  $(t, 1, \infty)$  is in  $\mathbb{D}(\Delta_\gamma^m, Q)$ , ie  $t > n - 2m > \frac{n}{2}$ , then  $cf = -\Delta^m u + u^2$  would be in  $B_{1,\infty}^{t-2m}$ , which by Proposition 2.10 would imply  $t \leq k+a+2m \leq \frac{n}{2}$ , giving a contradiction. Hence the ill-definedness of  $Q$  at  $B_{1,\infty}^{k+a+2m}$  is not explained by partial knowledge at  $p = 1$ , but rather by the fact that  $Q$  is defined on  $H^m \ni u$ .

All in all there are legion examples of regularity properties corresponding to spaces outside of the parameter domains of  $A$ -moderacy. They are of importance for the general theory of partial differential equations, albeit at some distance from the most common boundary problems of mathematical physics.

**8.2. Other types of problems.** The analysed product type operators are obtained roughly by inserting derivatives of the unknown  $u$  in a polynomial of degree two; cf Section 5. This restriction to the second order case could seem artificial, but it has been made in order not to burden the exposition.

In fact products  $u_1(x) \dots u_m(x)$  have been analysed in  $B_{p,q}^s$  and  $F_{p,q}^s$  spaces by paramultiplication in eg [RS96, Ch. 4.5]. The approach is the same as for  $m = 2$  with collection of terms in two groups to which the dyadic ball and corona criteria applies, respectively, but the complexity of this is rather larger for  $m > 2$  because of the many indices. When needed one can undoubtedly obtain, say  $u^m = -L_u(u)$  and analyse the exact parilinearisation  $L_u$  along the lines of Theorem 5.7, using the framework of [RS96, Ch. 4.5]. Therefore these applications are left for the future, while the second order case is treated here with its consequences for eg the von Karman problem in Section 6; as mentioned the developed results also apply to the stationary Navier–Stokes equation.

An extension of the parametrix formulae to quasi-linear problems seems to require further techniques.

*Remark 8.3.* Parabolic boundary value problems could also be covered by Theorem 3.2, by taking  $A$  as the full parabolic system  $(\partial_t - a(x, D_x), r_0, T)$  acting in anisotropic spaces ( $r_0$  is restriction to  $t = 0$ , and  $T$  a trace operator defining the boundary conditions). For the linear problems, the reader is referred to [Gru95, Sect. 4] for the  $L_p$ -theory (using classical Besov and Bessel potential spaces) with a complete set of compatibility conditions on fully inhomogeneous data. In particular Corollary 4.5 there applies because the underlying manifold  $]0, b[ \times \Omega$  for  $0 < b < \infty$  is bounded, so that the solution spaces  $X_p^s$  fulfil (I) above. Because of the stronger data norms introduced to control the compatibility of the boundary- and initial-data for exceptional values of  $s$ , cf [Gru95, (4.16)], it is here convenient that the  $Y_p^s$ -scale is not required to fulfil (3.1)–(3.3). (The compatibility conditions may force one to work with rather small parameter domains, once data are given. But even so the present results may well allow considerable improvements of the solution's integrability.) For the non-linear terms, the product type operators of Section 4 are straightforward to treat in the corresponding anisotropic spaces, since the necessary paramultiplication estimates have been established in this framework [Yam86a, Joh95].

For problems of composition type, T. Runst and the author [JR97] obtained solutions using the Leray–Schauder fixed point theorem and carried the existence over to a large domain of  $B_{p,q}^s$ - and  $F_{p,q}^s$ -spaces with a boot-strap argument. However, the domain of  $A$ -moderacy  $\mathbb{D}(\mathcal{A}, Q)$  is not convex for such problems, cf [JR97, Fig 1], so the iteration almost developed into a formal algorithm. J.-Y. Chemin and C.-J. Xu [CX97] used a boot-strap method to give a simplified proof of the smoothness of weak solutions to the Euler–Lagrange equations of harmonic maps; the basic step was to obtain hypoellipticity of a class of semi-linear problems with terms of the form  $\sum a_{j,k}(x, u(x)) \partial_j u \partial_k u$ . Formally this incorporates both composition and product type non-linearities, but the difficulties met in [JR97] did not show up in [CX97], since the weak solutions in this case are known to be bounded (so that consideration of  $u \mapsto F(u)$  on the *full* spaces  $H_p^s$  or  $B_{p,q}^s$  with  $1 < s < \frac{n}{p}$  was unnecessary). However, this well indicates that larger families of non-linearities will be relevant and potentially require disturbingly many additional efforts.

It has therefore been natural to treat only the class of product type operators in the present article, although Section 3 applies at least to bounded solutions of composition type problems. But the latter sphere of problems could in general deserve stronger methods, say to get rid of the boot-strap algorithm in [JR97]. However, it seems rather demanding to analyse the exact parilinearisation of  $F(u)$  when  $u$  is an unbounded function, say an element of  $H_p^s$  for  $s < \frac{n}{p}$ ; this is probably an open problem.

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